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4/4/16. Find, with proof, all integers n such that there is a solution in nonnegative real numbers (x, y, z) to the system of equations

$$2x^2 + 3y^2 + 6z^2 = n \quad \text{and} \quad 3x + 4y + 5z = 23.$$

Credit This problem was created by USAMTS Director Richard Rusczyk.

Comments There are a variety of solutions to this problem. Zhou Fan and Nathan Pflueger present algebraic approaches. Lawrence Chan considers a geometric interpretation of the problem, and Richard McCutchen uses vectors.

Solution 1 by: Zhou Fan (11/NJ)

Let $3x = x_1$, $4y = y_1$, and $5z = z_1$. Then $x_1 + y_1 + z_1 = 23$ and $\frac{2}{9}x_1^2 + \frac{3}{16}y_1^2 + \frac{6}{25}z_1^2 = n$. We can find bounds for n as follows:

For the upper bound, we note:

$$\begin{aligned} n &= \frac{2}{9}x_1^2 + \frac{3}{16}y_1^2 + \frac{6}{25}z_1^2 \\ &\le \frac{6}{25}(x_1^2 + y_1^2 + z_1^2) \\ &\le \frac{6}{25}(x_1 + y_1 + z_1)^2 \\ &= \frac{3174}{25} \end{aligned}$$

We can obtain equality when $(x_1, y_1, z_1) = (0, 0, 23)$.

For the lower bound, we note that $\frac{207}{28} + \frac{184}{21} + \frac{575}{84} = 23$, so we can let $(x_1, y_1, z_1) = (\frac{207}{28} + a, \frac{184}{21} + b, \frac{575}{84} + c)$ where $a + b + c = 0$. Then

$$\begin{aligned} n &= \frac{2}{9}x_1^2 + \frac{3}{16}y_1^2 + \frac{6}{25}z_1^2 \\ &= \frac{2}{9}\left(\frac{207}{28} + a\right)^2 + \frac{3}{16}\left(\frac{184}{21} + b\right)^2 + \frac{6}{25}\left(\frac{575}{84} + c\right)^2 \\ &= \frac{2}{9}a^2 + \frac{23}{7}a + \frac{13}{16}b^2 + \frac{23}{7}b + \frac{6}{25}c^2 + \frac{23}{7}c + \frac{529}{14} \\ &= \frac{2}{9}a^2 + \frac{13}{16}b^2 + \frac{6}{25}c^2 + \frac{529}{14} \\ &\ge \frac{529}{14} \end{aligned}$$

We can obtain equality when $a, b, c = 0$ and thus $(x_1, y_1, z_1) = (\frac{207}{28}, \frac{184}{21}, \frac{575}{84})$.

We note that the ranges of values of x_1, y_1, and z_1 are all continuous intervals since the original x, y, z can be any nonnegative real numbers. Thus, we can have a continuous change between $(x_1, y_1, z_1) = (\frac{207}{28}, \frac{184}{21}, \frac{575}{84})$ and $(x_1, y_1, z_1) = (0, 0, 0)$, so n can take any real value between $\frac{529}{14}$ and $\frac{3174}{25}$. Thus, if n is restricted to the integers, then n can be any integer between 38 and 126 inclusive.

Solution 2 by: Nathan Pflueger (12/WA)

We shall show that all integers n, $38 \leq n \leq 126$, there exist nonnegative reals (x, y, z) such that $2x^2 + 3y^2 + 6z^2 = n$ and $3x + 4y + 5z = 23$, and these are the only integers n with this characteristic.

Define $f(x, y, z) = 2x^2 + 3y^2 + 6z^2$, for any nonnegative reals x, y, z, satisfying $3x + 4y + 5z = 23$. Note that "$f(x, y, z) = a$" carries both the implication that $2x^2 + 3y^2 + 6z^2 = a$ as well as $3x + 4y + 5z = 23$, f only being defined on its given domain.

LEMMA: If $f(x_1, y_1, z_1) = a_1$, and $f(x_2, y_2, z_2) = a_2$, then for all a between a_1 and a_2, there exist x, y, z such that $f(x, y, z) = a$.
PROOF: Define the function $g(t) = f(x_1 + (x_2 - x_1)t, y_1 + (y_2 - y_1)t, z_1 + (z_2 - z_1)t)$. The arguments given for f lie between the (x_1, y_1, z_1) and (x_2, y_2, z_2), so they are nonnegative, and they satisfy $3x + 4y + 5z = 23$, so g is defined for all $t \in [0, 1]$. g expands out to a polynomial function of t, so it is continuous. Since $g(0) = a_1$ and $g(1) = a_2$, for all a between a_1 and a_2 there is some $t \in [0, 1]$ such that $g(t) = a$, and thus some (x, y, z) such that $f(x, y, z) = a$.

By the Cauchy-Schwarz inequality, $(x\sqrt{2} \cdot \frac{3}{\sqrt{2}} + y\sqrt{3} \cdot \frac{4}{\sqrt{3}} + z\sqrt{6} \cdot \frac{5}{\sqrt{6}})^2 \leq (2x^2 + 3y^2 + 6z^2)(\frac{9}{2} + \frac{16}{3} + \frac{25}{6}) = 14 \cdot f(x, y, z)$, so $f(x, y, z) \geq \frac{1}{14}(3x + 4y + 5z)^2 = \frac{23^2}{14} = \frac{529}{14}$. By applying the equality condition of Cauchy, we find that $f(\frac{69}{28}, \frac{46}{21}, \frac{115}{84}) = \frac{529}{14}$.

By the trivial ineqality and the fact that $\frac{3}{16} < \frac{2}{9} < \frac{6}{25}$, $f(x, y, z) = \frac{2}{9}(3x)^2 + \frac{3}{16}(4y)^2 + \frac{6}{25}(5z)^2 \leq \frac{6}{15}((3x)^2 + (4y)^2 + (5z)^2)$. Since $x, y, z > 0$ and due to the expansion of $(a+b+c)^2$, this is in turn less than or equal to $\frac{6}{15}(3x + 4y + 5z)^2 = \frac{6}{15}23^2 = \frac{3174}{25}$. Thus we have $f(x, y, z) \leq \frac{3174}{25}$. The equality case is achievable; $f(0, 0, \frac{23}{5}) = \frac{3174}{25}$.

Thus we have determined that $\frac{529}{14} \leq f(x, y, z) \leq \frac{3174}{25}$, and there exist x, y, z satisfying both equality cases. Therefore, by the lemma, for any integer $n \in [\frac{529}{14}, \frac{3174}{25}]$, there exist nonnegative reals (x, y, z) such that $f(x, y, z) = n$. $\frac{529}{14} = 37\frac{11}{14}$, and $\frac{3174}{25} = 126\frac{24}{25}$, so all possible *integer* values of n are the integers from 38 to 126.

Solution 3 by: Lawrence Chan (11/IL)

There is an interesting geometric solution to this problem. We begin by noting that $2x^2+3y^2+6z^2=n$ is an ellipsoid and $3x+4y+5z=23$ is a plane. If the ellipsoid becomes too small, it will not be able to touch the plane. If it becomes too big, its intersection with the plane will be outside the set of triples of positive reals. We will begin by finding the lower bound.

Since we want to find the point where the ellipsoid just touches the plane, we want to find n such that the ellipsoid is tangent to the plane. To simplify things, we will rearrange variables and define new variables.

$$\begin{aligned} 2x^2+3y^2+6z^2 &= n \\ (\sqrt{2}x)^2+(\sqrt{3}y)^2+(\sqrt{6}z)^2 &= n \end{aligned}$$

We will define

$$\begin{aligned} u &:= \sqrt{2}x \\ v &:= \sqrt{3}y \\ w &:= \sqrt{6}z \end{aligned}$$

Our equation then becomes

$$u^2+v^2+w^2=n$$

which is a sphere centered at (0,0,0) with radius $\sqrt{n}$.

We will also change the variables in the equation of the plane to get

$$\frac{3}{\sqrt{2}}u+\frac{4}{\sqrt{3}}v+\frac{5}{\sqrt{6}}w=23$$

If a plane is tangent to a sphere centered at the origin, the distance from the plane to the origin is equal to the radius of the sphere. Thus,

$$\begin{aligned} \sqrt{n} &= \frac{|\frac{3}{\sqrt{2}}(0)+\frac{4}{\sqrt{3}}(0)+\frac{5}{\sqrt{6}}(0)-23|}{\sqrt{(\frac{3}{\sqrt{2}})^2+(\frac{4}{\sqrt{3}})^2+(\frac{5}{\sqrt{6}})^2}} \\ n &= \frac{23^2}{14} \\ n &\approx 37.7857 \end{aligned}$$

We want the the first n that works, so our lower bound is $n=38$. To find the upper bound, we check the axes since those are the farthest reaching (in other words, we let x and y equal 0 to find the upper bound by z, and we do this for each of the three variables). We want to find the largest n such that

$$\begin{aligned}
\sqrt{\frac{n}{6}} &= \frac{23}{5} \text{ checking z axis} \\
n &\approx 126.96 \\
\sqrt{\frac{n}{3}} &= \frac{23}{4} \text{ checking y axis} \\
n &\approx 99.19 \\
\sqrt{\frac{n}{2}} &= \frac{23}{3} \text{ checking x axis} \\
n &\approx 117.56
\end{aligned}$$

The highest n that still works is 126, so our answer is as follows:

$\boxed{\text{n}=\{x : x \in \mathbf{Z} \text{ and } 38 \leq x \leq 126\}}$

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Solution 4 by: Richard McCutchen (10/MD)

A change of variables will make this problem a bit easier. Let $x = u/\sqrt{2}$, $y = v/\sqrt{3}$, and $z = w/\sqrt{6}$; the new variables u, v, and w are nonnegative iff x, y, and z are. Our two equations become

$$u^2 + v^2 + w^2 = n, \tag{1}$$

$$\frac{3}{\sqrt{2}}u + \frac{4}{\sqrt{3}}v + \frac{5}{\sqrt{6}}w = 23. \tag{2}$$

Introduce a rectangular uvw coordinate system; we consider only the region where $u, v, w \geq 0$ (henceforth the first octant). (1) asserts that the distance from (u, v, w) to the origin is $\sqrt{n}$, while (2) asserts that (u, v, w) is on a plane (call it $\mathcal{P}$) that does not depend on n. Thus, if we can determine at what distances the first-octant points on $\mathcal{P}$ lie from the origin, we'll know for which values of n the system is solvable.

For convenience, let

$$\vec{\mathbf{a}} = \left\langle \frac{3}{\sqrt{2}}, \frac{4}{\sqrt{3}}, \frac{5}{\sqrt{6}} \right\rangle \qquad \text{and} \qquad \vec{\mathbf{b}} = \langle u, v, w \rangle .$$

We can then rewrite (1) and (2) as

$$|\vec{\mathbf{b}}| = \sqrt{n}, \tag{1$'$}$$

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 23. \tag{2$'$}$$

It is also useful to know that $|\vec{\mathbf{a}}| = \sqrt{14}$:

$$\left|\left\langle \frac{3}{\sqrt{2}}, \frac{4}{\sqrt{3}}, \frac{5}{\sqrt{6}} \right\rangle\right| = \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{4}{\sqrt{3}}\right)^2 + \left(\frac{5}{\sqrt{6}}\right)^2} = \sqrt{\frac{9}{2} + \frac{16}{3} + \frac{25}{6}} = \sqrt{14}.$$

The first-octant point of $\mathcal{P}$ nearest the origin

The Cauchy-Schwartz Inequality on vectors states that

$$(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}) \leq |\vec{\mathbf{a}}| \cdot |\vec{\mathbf{b}}|, \tag{3}$$

where $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are as above. I'd like to use this to obtain a lower bound on the distance from points $\vec{\mathbf{b}}$ on $\mathcal{P}$ from the origin; this distance is $|\vec{\mathbf{b}}|$. Since $\vec{\mathbf{b}}$ is on $\mathcal{P}$, we can substitute for the left side using $(2')$, and we already know what $|\vec{\mathbf{a}}|$ is:

$$23 \leq \sqrt{14}|\vec{\mathbf{b}}| \Rightarrow |\vec{\mathbf{b}}| \geq 23/\sqrt{14}. \tag{4}$$

This is interesting.

There is, in fact, a point of $\mathcal{P}$ in the first octant that lies at a distance of $23/\sqrt{14}$ from the origin. It is $\vec{\mathbf{b}} = (23/14)\vec{\mathbf{a}}$. This point lies on $\mathcal{P}$ because $(23/14)\vec{\mathbf{a}} \cdot \vec{\mathbf{a}} = (23/14)|\vec{\mathbf{a}}|^2 = 23$, and $|\vec{\mathbf{b}}| = (23/14)|\vec{\mathbf{a}}| = 23/\sqrt{14}$.

The first-octant point of $\mathcal{P}$ furthest from the origin

What first-octant point of $\mathcal{P}$ is furthest from the origin? To find this out, I would like to prove a lemma.

Lemma. Let $\mathcal{A} = A_1A_2 \ldots A_k$ be a polygon lying in 3-space, and let P be a point in 3-space. Of all the points on or inside $\mathcal{A}$, the point furthest from P is one of the vertices A_i. (If multiple points on or inside $\mathcal{A}$ tie for furthest, they are all vertices of $\mathcal{A}$.)

Proof. Let Q be any point of $\mathcal{A}$ other than a vertex; I will prove that there is a point of $\mathcal{A}$ further from P than Q. There must be a line segment $\overline{UV}$ lying on or inside $\mathcal{A}$ and having Q as its midpoint. (If Q is on a side of $\mathcal{A}$, take a short segment along this side. Otherwise, any sufficiently short segment with midpoint Q and in the plane of $\mathcal{A}$ will do.)

Let $\vec{\mathbf{w}} = \vec{QV} = \vec{UQ}$. Suppose $\vec{\mathbf{w}} \cdot \vec{PQ} \geq 0$. Then V is further from P than Q is, because

$$PV^2 = |\vec{PQ} + \vec{\mathbf{w}}|^2 = (\vec{PQ} + \vec{\mathbf{w}}) \cdot (\vec{PQ} + \vec{\mathbf{w}}) = PQ^2 + |\vec{\mathbf{w}}|^2 + 2PQ|\vec{\mathbf{w}}| > PQ^2 + 2PQ|\vec{\mathbf{w}}| \geq PQ^2.$$

(The last inequality follows because $\vec{\mathbf{w}} \cdot \vec{PQ} \geq 0$.) If $\vec{\mathbf{w}} \cdot \vec{PQ} < 0$, the proof is similar; U is further from P than Q is. Either way, we have proved the lemma. $\square$

It should be clear that the intersection of the fixed plane $\mathcal{P}$ with the first octant is a triangle (call it T) with one vertex on each axis. Using (2), these vertices are

$$\left(\frac{23\sqrt{2}}{3}, 0, 0\right), \quad \left(0, \frac{23\sqrt{3}}{4}, 0\right), \quad \left(0, 0, \frac{23\sqrt{6}}{5}\right).$$

A quick calculator approximation shows that $(0, 0, 23\sqrt{6}/5)$ is the furthest from the origin among these three. The lemma then says that $(0, 0, 23\sqrt{6}/5)$ is furthest from the origin among all points of T. Of course, the distance is $23\sqrt{6}/5$.

Putting it together

So, we know that, of the points of the triangle T, $(23/14)\vec{\mathbf{a}}$ ($\vec{\mathbf{a}}$ was the messy vector from a while ago) is the closest to the origin at a distance of $23\sqrt{14}$, and $(0, 0, 23\sqrt{6}/5)$ is the furthest from the origin at a distance of $23\sqrt{6}/5$. Now, back to the system

$$u^2 + v^2 + w^2 = n, \tag{1}$$

$$\frac{3}{\sqrt{2}}u + \frac{4}{\sqrt{3}}v + \frac{5}{\sqrt{6}}w = 23. \tag{2}$$

(2) says that (u, v, w) is on or inside T. (1) says that (u, v, w) is at a distance $\sqrt{n}$ from the origin. Thus, the system is solvable for a given n iff T has a point $\sqrt{n}$ units from the origin. Thus, if n could be a real number, it could range from

$$(23/\sqrt{14})^2 = 529/14 \approx 37.78 \quad \text{to} \quad (23\sqrt{6}/5)^2 = 3174/25 = 126.96.$$

Since n must be an integer, the system is solvable for $38 \leq n \leq 126$. ■