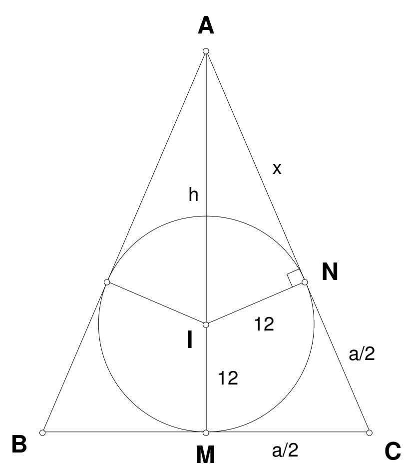
2/3/16. Find three isosceles triangles, no two of which are congruent, with integer sides, such that each triangle's area is numerically equal to 6 times its perimeter.

Credit This is a slight modification of a problem provided by Suresh T. Thakar of India. The original problem asked for five isosceles triangles with integer sides such that the area is numerically 12 times the perimeter.

Comments Many students simply set up an equation using Heron's formula and then turned to a calculator or a computer for a solution. Below are presented more elegant solutions. Zachary Abel shows how to reduce this problem to finding Pythagorean triples which have 12 among the side lengths. Adam Hesterberg gives us a solution using Heron's formula. Finally, Kristin Cordwell shows how to take an intelligent trial-and-error approach to construct the solutions. *Solutions edited by Richard Rusczyk.*

Solution 1 by: Zachary Abel (11/TX)

In $\triangle ABC$ with $AB = AC$, we use the common notations $r =$ inradius, $s =$ semiperimeter, $p =$ perimeter, $K =$ area, $a = BC$, and $b = AC$. The diagram shows triangle ABC with its incircle centered at I and tangent to BC and AC at M and N respectively.



The area of the triangle is given by $rs = K = 6p = 12s$, which implies $r = 12$. The area is also

$$K = \frac{a}{2} \cdot AM = \frac{a}{2} \cdot \sqrt{AC^2 - MC^2} = \frac{a}{4}\sqrt{4b^2 - a^2}.$$

Since K is an integer (since it is six times the perimeter), $4b^2 - a^2$ must be a perfect square. If a were odd, then $4b^2 - a^2 \equiv 3 \bmod 4$, which is not possible for a perfect square. Thus, a is even. So $x = AN = b - \frac{a}{2}$ is an integer, and so is $h = AM - 12 = \frac{1}{2}\sqrt{4b^2 - a^2} - 12$. So, since ANI is a right triangle, the integers x, 12, and h form a Pythagorean triple.

It is easy to check that 12 can be the leg in only four Pythagorean triangles: $(5, 12, 13)$, $(12, 16, 20)$, $(9, 12, 15)$, and $(12, 35, 37)$. So these give all the possibilities for x and h.

Similar triangles ANI and AMC show that $\frac{x}{12} = \frac{h+12}{(a/2)}$, i.e. $a = \frac{24(h+12)}{x}$, which gives $a = 120, 48, 72, \frac{168}{5}$ for the four respective cases. Since a is an integer, the fourth case doesn't work. Now, since $b = x + \frac{a}{2}$, we find that $b = 65, 40, 45$ respectively for the three remaining cases. Therefore, the three triangles have dimensions $(120, 65, 65)$, $(48, 40, 40)$, and $(72, 45, 45)$, and there are no others that satisfy the conditions of the problem.

Solution 2 by: Adam Hesterberg (10/WA)

Answers: Triangles with sides (72,45,45), (48,40,40), (120,65,65).

Let the sides of the triangle be (a, b, b). Then the six times the perimeter of the triangle is $6a + 12b$, and the area of the triangle, by Heron's formula, is $\sqrt{\left(b+\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(b-\frac{a}{2}\right)}$.

$$\begin{aligned}
6a + 12b &= \sqrt{\left(b+\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(b-\frac{a}{2}\right)} \\
144\left(b+\frac{a}{2}\right)^2 &= \left(\frac{a}{2}\right)^2\left(b-\frac{a}{2}\right)\left(b+\frac{a}{2}\right) \\
144b + 72a &= \left(\frac{a}{2}\right)^2\left(b-\frac{a}{2}\right) \\
\frac{2b+a}{2b-a} &= \left(\frac{a}{24}\right)^2
\end{aligned}$$

Trying multiples of 24 for a leads to ($a = 48$, $b = 40$), ($a = 72$, $b = 45$), and ($a = 120$, $b = 65$). These have areas of 972, 768, and 1500, respectively, all of which are 6 times their perimeters.

Solution 3 by: Kristin Cordwell (8/NM)

Three isosceles triangles whose area equals six times their perimeter are 45 by 45 by 72, 40 by 40 by 48, and 65 by 65 by 120.

To begin with, we notice that two congruent right triangles stuck together at a common leg form an isosceles triangle. We then consider common right triangles: 3,4,5 and 5,12,13. The other feature that we need to note is that, if we scale the perimeter by a factor α, then the area scales by α^2.

If we take two 3,4,5 right triangles and join them along the short side, we get an isosceles triangle of $P = 18$, and $A = 12$. Since $P = 2 \cdot 3^2$ and $A = 2^2 \cdot 3$, we see that, if we scale P by 3, A will scale by 3^2, and they will be in the proportion $A :: P = 2^2 \cdot 3^3 :: 2 \cdot 3^3$. This isn't quite what we want, but if we scale P by 3 one more time, we will end up with $2^2 \cdot 3^5 :: 2 \cdot 3^4$, or $A = 6P$.

If we take two 3,4,5 right triangles and join them along the long side, we get an isosceles triangle of $P = 16$, and $A = 12$. Note that we already have a factor of 3 in the A to P ratio, but that we need another net factor of 2^3 to get the overall ratio of 6. Since we "gain" a net factor of 2 for every doubling of the perimeter, we scale P by 8, which gives $P = 128$ and $A = 768 = 6P$.

Finally, if we consider two 5,12,13 triangles and glue them together at the short leg, we have a triangle of sides 13, 13, and 24, with $P = 50 = 2 \cdot 5^2$, and $A = 60 = 2^2 \cdot 3 \cdot 5$. The powers of 2 and 3 are what we wish, but we need to scale P by 5 (and A by 5^2, in order to have the powers of 5 balance. We then obtain $P = 250$ and $A = 1500 = 6P$.

In some cases, we can scale by a fraction. For example, if we look at two 7,24,25 right triangles joined at the short edge, we have $P = 98 = 2 \cdot 7^2$ and $A = 168 = 2^3 \cdot 3 \cdot 7$. If we scale P by $\frac{7}{2}$, we get a new perimeter of $P = 7^3$ and an area $A = 2 \cdot 3 \cdot 7^3$. This works because we have a starting power of 2 in the perimeter that can be canceled.