USA Mathematical Talent Search

Round 3 Problems

Year 34 — Academic Year 2022-2023

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name, username, and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

4. Submit your solutions by **January 3, 2023** via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 10 PM Eastern / 7 PM Pacific on January 3, 2023.

   (b) Mail: USAMTS
   55 Exchange Place
   Suite 603
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before January 3, 2023.

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging on to www.usamts.org and visiting the "My USAMTS" pages.

7. Round 3 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules at www.usamts.org.

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Each problem is worth 5 points.

1/3/34. In the 8×8 grid below, label 8 squares with X and 8 squares with Y such that:

1. No square can be labeled with both an X and a Y.
2. Each row and each column must contain exactly one square labeled X and one square labeled Y.
3. Any square marked with a $\star$ or a $\heartsuit$ cannot be labeled with an X or a Y.
4. We say that a square marked with a $\star$ or a $\heartsuit$ *sees* a label (X or Y) if one can move in a straight line horizontally or vertically from the marked square to the square with the label, without crossing any other squares with X's or Y's. It is OK to cross other squares marked with a $\star$ or $\heartsuit$. Using this definition:
   (a) Each square marked with a $\star$ must see exactly 2 X's and 1 Y.
   (b) Each square marked with a $\heartsuit$ must see exactly 1 X and 2 Y's.

		$\star$	$\star$	$\star$	$\star$	$\star$	
							$\star$
							$\star$
	$\heartsuit$						$\star$
							$\star$
							$\heartsuit$
				$\star$			

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/3/34. Let $\mathbb{Z}^+$ denote the set of positive integers. Determine, with proof, if there exists a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$f(f(f(f(f(n))))) = 2022n$$

for all positive integers n.

3/3/34. A positive integer N is called *googolicious* if there are exactly 10^{100} positive integers x that satisfy the equation

$$\left\lfloor \frac{N}{\left\lfloor \frac{N}{x} \right\rfloor} \right\rfloor = x,$$

where $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z. Find, with proof, all googolicious integers.

4/3/34. Let ω be a circle with center O and radius 10, and let H be a point such that $OH = 6$. A point P is called *snug* if, for all triangles ABC with circumcircle ω and orthocenter H, we have that P lies on $\triangle ABC$ or in the interior of $\triangle ABC$. Find the area of the region consisting of all snug points.

5/3/34. A lattice point is a point on the coordinate plane with integer coefficients. Prove or disprove: there exists a finite set S of lattice points such that for every line ℓ in the plane with slope 0, 1, -1, or undefined, either ℓ and S intersect at exactly 2022 points, or they do not intersect.

Problems 1-4 by USAMTS staff. Problem 5 appeared in another form on Tournament of Towns 2020. The authors were Nikolai Beluhov, Atanas Dinev, and Kostadin Garov.
Round 3 Solutions must be submitted by **January 3, 2023**.
Please visit **http://www.usamts.org** for details about solution submission.