USA Mathematical Talent Search

Round 3 Problems

Year 32 — Academic Year 2020-2021

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name, username, and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

4. Submit your solutions by **January 5, 2021** via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 10 PM Eastern / 7 PM Pacific on January 5, 2021.

   (b) Mail: USAMTS
   55 Exchange Place
   Suite 603
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before January 5, 2021.

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging on to www.usamts.org and visiting the "My USAMTS" pages.

7. Round 3 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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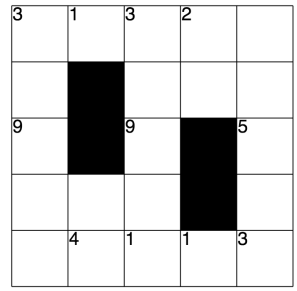
Each problem is worth 5 points.

1/3/32. Place the 21 two-digit prime numbers in the white squares of the grid on the right so that each two-digit prime is used exactly once. Two white squares sharing a side must contain two numbers with either the same tens digit or ones digit. A given digit in a white square must equal at least one of the two digits of that square's prime number.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: in any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/3/32. Find distinct points A, B, C, and D in the plane such that the length of the segment AB is an even integer, and the lengths of the segments AC, AD, BC, BD, and CD are all odd integers. In addition to stating the coordinates of the points and distances between points, please include a brief explanation of how you found the configuration of points and computed the distances.

3/3/32. Find, with proof, all positive integers n with the following property: There are only finitely many positive multiples of n which have exactly n positive divisors.

4/3/32. In a group of $n > 20$ people, there are some (at least one, and possibly all) pairs of people that know each other. Knowing is symmetric; if Alice knows Blaine, then Blaine also knows Alice. For some values of n and k, this group has a peculiar property: If any 20 people are removed from the group, the number of pairs of people that know each other is at most $\frac{n-k}{n}$ times that of the original group of people.

(a) If $k = 41$, for what positive integers n could such a group exist?

(b) If $k = 39$, for what positive integers n could such a group exist?

5/3/32. Let $n \geq 3$ be an integer. Let f be a function from the set of all integers to itself with the following property: If the integers $a_1, a_2, \ldots, a_n$ form an arithmetic progression, then the numbers

$$f(a_1), f(a_2),\ \ldots,\ f(a_n)$$

form an arithmetic progression (possibly constant) in some order. Find all values for n such that the only functions f with this property are the functions of the form $f(x) = cx + d$, where c and d are integers.

Problems by David Altizio, Michael Tang, Kevin Ren, and USAMTS Staff.

Round 3 Solutions must be submitted by **January 5, 2021**.

Please visit **http://www.usamts.org** for details about solution submission.