USA Mathematical Talent Search

Round 3 Problems

Year 31 — Academic Year 2019-2020

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem.

4. Submit your solutions by **January 6, 2020** via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 8 PM Eastern / 5 PM Pacific on January 6, 2020.

   (b) Mail: USAMTS
   55 Exchange Place
   Suite 603
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before January 6, 2020.

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages.

7. Round 3 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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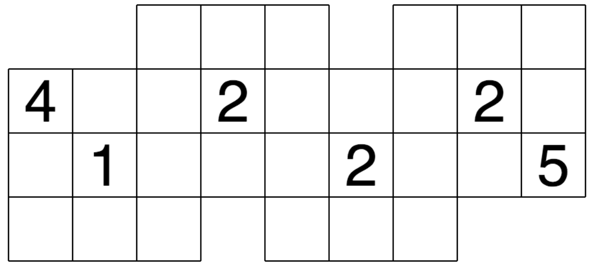
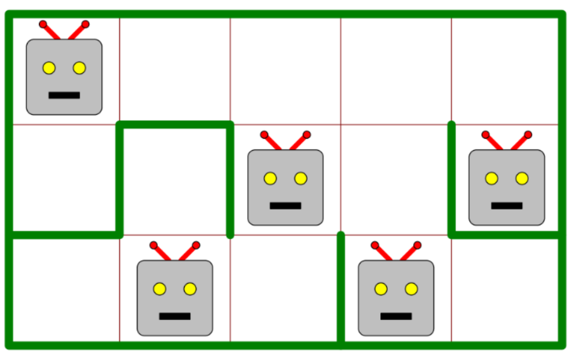
Each problem is worth 5 points.

1/3/31. Fill in each square with a number from 1 to 5; some numbers have been given. If two squares A and B have equal numbers, then A and B cannot share a side, and there also cannot exist a third square C sharing a side with both A and B.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/3/31. An apple orchard's layout is a rectangular grid of unit squares. Some pairs of adjacent squares have a thick wall of grape vines between them. The orchard wants to post some robot sentries to guard its prized apple trees. Each sentry occupies a single square of the layout, and from there it can guard both its square and any square in the same row and column that it can see, where only walls and the edges of the orchard block its sight. A sample layout (not the layout of the actual orchard, which is not given) is shown below.



Although a square may be guarded by multiple sentries, the sentries have not been programmed to avoid attacking other sentries. Thus, no sentry may be placed on a square guarded by another sentry. The orchard's expert has found a way to guard all the squares of the orchard by placing 1000 sentries. However, the contractor shipped 2020 sentries. Show that it is impossible for the orchard to place all 2020 of the sentries without two of them attacking each other.

3/3/31. A positive integer $n > 1$ is *juicy* if its divisors $d_1 < d_2 < \cdots < d_k$ satisfy $d_i - d_{i-1} | n$ for all $2 \leq i \leq k$. Find all squarefree juicy integers.

4/3/31. Let FIG be a triangle and let D be a point on $\overline{FG}$. The line perpendicular to $\overline{FI}$ passing through the midpoint of $\overline{FD}$ and the line perpendicular to $\overline{IG}$ passing through the midpoint of $\overline{DG}$ intersect at T. Prove that $FT = GT$ if and only if $\overline{ID}$ is perpendicular to $\overline{FG}$.

5/3/31. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for any reals x, y,

$$f(x+y)f(x-y) = (f(x))^2 - (f(y))^2.$$

Additionally, suppose that $f(x+2\pi) = f(x)$ and that there does not exist a positive real $a < 2\pi$ such that $f(x+a) = f(x)$ for all reals x. Show that for all reals x,

$$\left| f\left(\frac{\pi}{2}\right) \right| \geq f(x).$$