1/2/33. A 5×5 **Latin Square** is a 5×5 grid of squares in which each square contains one of the numbers 1 through 5 such that every number appears exactly once in each row and column. A partially completed grid (with numbers in some of the squares) is **puzzle-ready** if there is a unique way to fill in the remaining squares to complete a Latin Square.

Below is a partially completed grid with seven squares filled in and an additional three squares shaded. Determine what numbers must be filled into the shaded squares to make the grid (now with ten squares filled in) puzzle-ready, and then complete the Latin Square.

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

1				
	1	3		
2	3			5
			5	

Solution

1	4	5	2	3
5	1	3	4	2
2	3	4	1	5
3	2	1	5	4
4	5	2	3	1

2/2/33. Let n be a fixed positive integer. Which is greater?

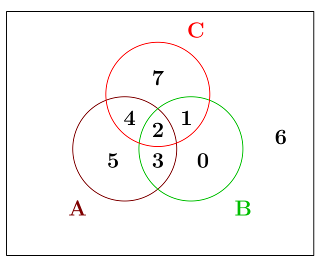
1. The number of n-tuples of integers whose largest value is 7 and whose smallest value is 0; or
2. The number of ordered triples (A, B, C) that satisfy the following property: A, B, C are subsets of $\{1, 2, 3, \dots, n\}$, and neither $C \subseteq A \cup B$, nor $B \subseteq A \cup C$.

Your answer can be: (1), (2), the two counts are equal, or it depends on n.

Solution

The answer is that the two sets are **the same** size. This can be proven by directly counting each and comparing the results. Here, we will demonstrate a bijection between the two sets.

Note that sets A, B, and C break $\{1, 2, 3, \dots, n\}$ up into 8 total regions. We will label each element in the set based on which region it is in. We will call the region in B but not in A or C region 0. The region in C but not in A or B will be region 7. The other regions will be labeled arbitrarily.



Given this Venn Diagram, any n-tuple of numbers from 0 to 7 can be used to construct sets A, B, and C. We simply put the k-th element in the region defined by the number assigned to it. So, for example, if the first element were labeled 1, we would put it in both sets B and C but not in A.

Now, let's look at conditions **(1)** and **(2)** in the problem statement.

We'll start with **(1)**. If an assigned n-tuple has a maximum of 7, then that means the corresponding subsets have at least one element that is in C but not in A or B. That means $C \not\subseteq A \cup B$. Similarly, if it has a minimum of 0, then that means the corresponding subsets have an element in B that is not in A or C. That means $B \not\subseteq A \cup C$. Hence, every n-tuple from part **(1)** will map to A, B, and C with the properties in **(2)**.

Likewise, if a set of subsets satisfies **(2)**, then since $B \not\subseteq A \cup C$, that means region 0 is occupied, or at least one number is assigned to region 0. Hence, our n-tuple from A will use the 0. Similarly, since $C \not\subseteq A \cup B$, we know that there is at least one number in region 7.

Thus, we have established a bijection and we see that **(1)** and **(2)** have the same number of elements! Although, of course, this solution makes absolutely no claim as to what that number of elements is! We'll leave that to you!

□

3/2/33. Let x and y be distinct real numbers such that

$$\sqrt{x^2+1}+\sqrt{y^2+1}=2021x+2021y.$$

Find, with proof, the value of

$$(x+\sqrt{x^2+1})(y+\sqrt{y^2+1}).$$

Solution

Subtracting $x+y$ from both sides, we get

$$\left(\sqrt{x^2+1}-x\right)+\left(\sqrt{y^2+1}-y\right)=2020(x+y).$$

Because $\left(\sqrt{a^2+1}-a\right)\left(a+\sqrt{a^2+1}\right)=(a^2+1)-a^2=1$ for all real a, we have

$$\frac{1}{x+\sqrt{x^2+1}}+\frac{1}{y+\sqrt{y^2+1}}=2020(x+y).$$

Writing the left-hand side under a common denominator, we get

$$\frac{(x+\sqrt{x^2+1})+\left(y+\sqrt{y^2+1}\right)}{(x+\sqrt{x^2+1})\left(y+\sqrt{y^2+1}\right)}=2020(x+y).$$

But, since $\sqrt{x^2+1}+\sqrt{y^2+1}=2021(x+y)$, we have

$$\frac{2022(x+y)}{(x+\sqrt{x^2+1})\left(y+\sqrt{y^2+1}\right)}=2020(x+y).$$

Because $\sqrt{x^2+1}+\sqrt{y^2+1}\geq 1+1>0$, we have $2021(x+y)>0$, so $x+y>0$. Thus, we may write

$$\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)=\frac{2022(x+y)}{2020(x+y)}=\boxed{\frac{1011}{1010}}.$$

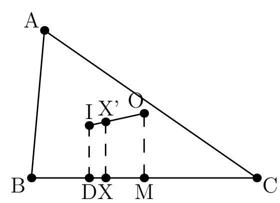
□

4/2/33. Let ABC be a scalene triangle, and let X, Y, Z be points on sides BC, CA, AB, respectively. Let I and O denote the incenter and circumcenter, respectively, of triangle ABC. Suppose that

$$\frac{BX - CX}{BA - CA} = \frac{CY - AY}{CB - AB} = \frac{AZ - BZ}{AC - BC}.$$

Prove that there exists a point P on line IO such that $\overline{PX} \perp \overline{BC}$, $\overline{PY} \perp \overline{CA}$, and $\overline{PZ} \perp \overline{AB}$.

Solution



To avoid lengthy discussion of configurations, let us now take all ratios of lengths to be directed. Hence we may write the given condition as

$$\alpha = \frac{BX - XC}{|AB| - |AC|} = \frac{CY - YA}{|BC| - |BA|} = \frac{AZ - ZB}{|CA| - |CB|}.$$

Observe that if $\alpha = 0$, we may select $P \equiv O$, and if $\alpha = 1$, we may select $P \equiv I$. Thus it suffices to consider the case where $\alpha \neq 0, 1$.

Since ABC is scalene, the points I and O are distinct, and $\overline{IO}$ is not perpendicular to $\overline{BC}$. Let M be the midpoint of $\overline{BC}$ and let D be the tangency point of the incircle onto side $\overline{BC}$.

The perpendicular to $\overline{BC}$ to X is not parallel to line IO, so the two intersect at some point X'. Evidently

$$\frac{OX'}{OI} = \frac{MX}{MD}.$$

However, notice that

$$MX = \frac{CX - XB}{2}$$

and

$$MD = \frac{|AC| - |AB|}{2}$$

so

$$\frac{MX}{MD} = \alpha.$$

Hence if we define Y' and Z' similarly, we find that $X' \equiv Y' \equiv Z'$, so the required P exists and lies on line IO.

$\square$

5/2/33. For a finite nonempty set A of positive integers, $A = \{a_1, a_2, \ldots, a_n\}$, we say the **calamitous complement** of A is the set of all positive integers k for which there do ***not*** exist nonnegative integers $w_1, w_2, \ldots, w_n$ with

$$k = a_1w_1 + a_2w_2 + \cdots + a_nw_n.$$

The calamitous complement of A is denoted $cc(A)$. For example,

$$cc(\{5, 6, 9\}) = \{1, 2, 3, 4, 7, 8, 13\}.$$

Find all pairs of positive integers a, b with $1 < a < b$ for which there exists a set G satisfying all of the following properties:

1. G is a set of at most three positive integers,
2. $cc(\{a, b\})$ and $cc(G)$ are both finite sets, and
3. $cc(G) = cc(\{a, b\}) \cup \{m\}$ for some m not in $cc(\{a, b\})$.

Solution

The answers are $(a, b) = (2, 2k+1)$, $(a, b) = (3, 3k+1)$, or $(a, b) = (3, 3k+2)$ where k is any positive integer. It will suffice to prove that $a \in \{2, 3\}$ and $\gcd(a, b) = 1$.

We cite the well-known Frobenius Coin problem to assert that $cc(\{a, b\})$ is finite if and only if $\gcd(a, b) = 1$.

First, we show that the pairs we listed in our answer work. We will use $G = \{a, a+b, 2b\}$. It is clear that $cc(G) \subseteq cc(\{a, b\})$. We show that $cc(G) = cc(\{a, b\}) \cup \{b\}$. This means that for any $x, y \geq 0$, we need to write $xa + yb$ as a nonnegative linear combination of $a, a+b, 2b$ except the case $(x, y) = (0, 1)$. That $(x, y) = (0, 1)$ is impossible follows from the fact that out of $\{a, a+b, 2b\}$, only a is less than b, and $\gcd(a, b) = 1$.

If $y = 0$, then xa is a linear combination of as only. If $y = 1$ and $x > 0$, then we use 1 term $a + b$ and $x - 1$ terms a. So it remains to consider $y \geq 2$.

If $a = 2$, then $3b = 1(a+b) + (b-1)a$, and if $a = 3$, then $3b = ba$. Either way, this means $3b \notin cc(G)$. Next, observe that all multiples of b greater than b can be written as a sum of terms of the form $2b$ and $3b$. This means that all numbers of the form $xa+yb$ for $x \geq 0, y \geq 2$ are not in $cc(G)$. That finishes our proof of the claim that $cc(G) = cc(\{a, b\}) \cup \{b\}$.

Now we need to show no other pairs work. The Frobenius Coin problem tells us immediately that $\gcd(a,b) > 1$ fails, so it remains to prove that $a < 4$. Suppose for the sake of contradiction that $a, b \geq 4$ and that a set G exists satisfying the properties (1), (2), (3) for some a, b. Note that we will remove the restriction that $a < b$ for this argument.

First, we show that the m in property (3) is one of a or b. If m is neither of them, then since $a, b \notin cc(\{a,b\})$, we must have $a, b \notin cc(G)$ by (3). But then a, b are nonnegative linear combinations of elements of G, which means that $cc(\{a,b\}) \subseteq cc(G)$. This means that no such m can exist. This contradiction means that $m \in \{a,b\}$. Without loss of generality, we will assume $m = b$.

We now claim that two of the elements of G must be a and $a+b$. Let H be the set of positive integers not in $cc(G)$, which means H is all positive integers that are a nonnegative linear combination of elements of G. The smallest element in H not a multiple of b is a. Since $\gcd(a,b) = 1$, a cannot be a linear combination of smaller elements of H, so a must be in G. Likewise, a is the only element in H that is smaller than $a+b$ and not a multiple of b. Because b cannot be in H by assumption, this means $a+b$ must also be in G.

We now know that $G = \{a, a+b, x\}$ for some x. The smallest element of H that is not a nonnegative linear combination of $a, a+b$ is $2b$. This is because all elements of H are of the form $xa + yb$ for x, y nonnegative integers and $(x,y) \neq (0,1)$, and adding terms of the form $a, a+b$ will obtain all such numbers when y is 0 or 1. Furthermore, $2b$ is not a linear combination of $a, a+b$ since $\gcd(a,b) = 1$ and $a \geq 4$.

So we must have $G = \{a, a+b, 2b\}$. We now show $3b$ is not a linear combination of any elements of G, which obtains our final contradiction. $3b$ is not a sum of terms of the form $2b$, obviously. If we involve the other terms, then we have $xa + yb$ for $x > 0$. Since $\gcd(a,b) = 1$, this can only be a multiple of b if x is a multiple of b. So $x \geq b$, but $a \geq 4$ meaning that $xa + yb \geq 4b > 3b$, a contradiction.

$\square$