USA Mathematical Talent Search

Round 3 Solutions

Year 27 — Academic Year 2015–2016

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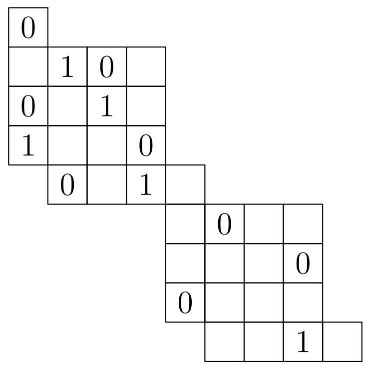
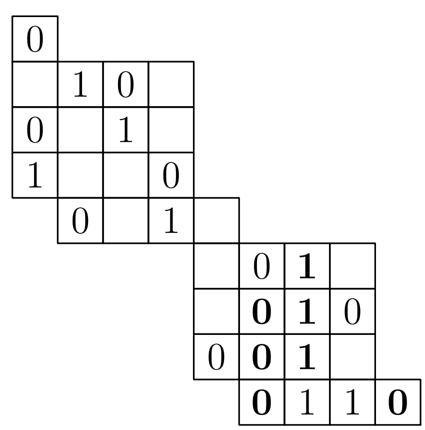
1/3/27. Fill in each space of the grid with either a 0 or a 1 so that all sixteen strings of four consecutive numbers across and down are distinct.



You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

Let row k be the k-th row with a horizontal word and let column j be the j-th column with a vertical word. (In particular, the lone "0" in the top left corner is *not* considered its own row.) We use the notation R_iC_j to denote the square in row i and column j. For example, we are given that R_1C_2 is a 1, while R_1C_1 is not given to us.



There are only two squares that intersect exactly one word, the top left and bottom right. Every other square is part of exactly two words. Therefore, if the bottom right square is a 1, the total number of 0's in all 16 strings is odd. But we know that there are 32 total 0's, so the bottom right square must be a 0.

This leaves column 7 as the only possible position for the string 1111. There are two possible locations for the string 0000: column 5 and column 6. However, if column 5 were 0000, then there would be 5 strings in the grid of the form $0x1x$ (rows 2, 4, 5, 6, and 7). But we know there are only 4 such strings, so the string 0000 must be placed in column 6.

0								
	1	0						
0		1						
1		**0**	0					
	0		1	**1**				
				1	0	1		
				1	0	1	0	
				0	0	1		
					0	1	1	0

Looking at the grid, we see that all strings of the form $0x1x$ are accounted for (rows 2, 4, 7, and 8). Thus, looking at column 3, we see that R_3C_3 is 0, looking at row 5, we see that R_5C_5 is 1, and looking at row 6, we see that R_6C_5 is 1. After placing these, looking at column 5 tells us that R_4C_5 is 1. Hence, column 5 is 1110.

Since row 6 is 1010, row 5 must be 1011. Since row 5 is 1011, column 8 must be 1001. Then column 5 and row 6 are both of the form $1x10$, which means R_3C_2 must be 0. Hence row 3 is 1000, and column 2 must be 1100. Since row 8 is 0110, we see that row 2 must be 0111, which means that row 4 must be 0011. Columns 2 and 3 are both of the form $x100$, so looking at row 1 tells us that R_1C_4 must be a 1. Finally, there are only 7 strings that start with a 0 in our grid so far, so R_1C_1 is a 0. This completes the grid.

0								
0	1	0	**1**					
0	**1**	1	**1**					
1	**0**	0	0					
	0	**0**	1	1				
				1	0	1	**1**	
				1	0	1	0	
				0	0	1	**0**	
					0	1	1	0

2/3/27. Fames is playing a computer game with falling two-dimensional blocks. The playing field is 7 units wide and infinitely tall with a bottom border. Initially the entire field is empty. Each turn, the computer gives Fames a 1×3 solid rectangular piece of three unit squares. Fames must decide whether to orient the piece horizontally or vertically and which column(s) the piece should occupy (3 consecutive columns for horizontal pieces, 1 column for vertical pieces). Once he confirms his choice, the piece is dropped straight down into the playing field in the selected columns, stopping all three of the piece's squares as soon as the piece hits either the bottom of the playing field or any square from another piece. All of the pieces must be contained completely inside the playing field after dropping and cannot partially occupy columns.

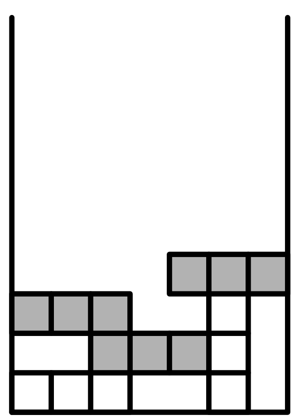
If at any time a row of 7 spaces is all filled with squares, Fames scores a point.

Unfortunately, Fames is playing in *invisible mode*, which prevents him from seeing the state of the playing field or how many points he has, and he already arbitrarily dropped some number of pieces without remembering what he did with them or how many there were.

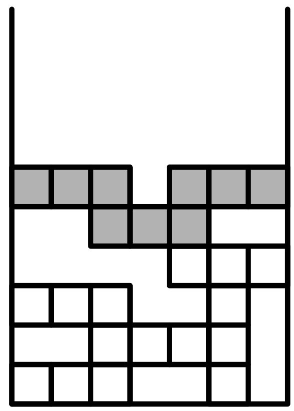
For partial credit, find a strategy that will allow Fames to eventually earn at least one more point. For full credit, find a strategy for which Fames can correctly announce "I have earned at least one more point" and know that he is correct.

Solution

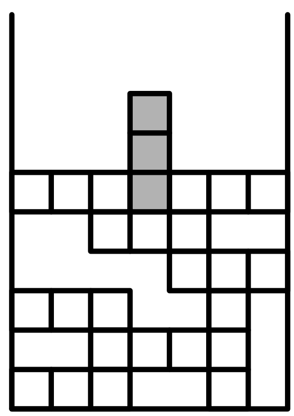
Drop a horizontal piece occupying columns 3, 4, and 5. Then drop two horizontal pieces occupying columns 1, 2, and 3 and columns 5, 6, and 7. We illustrate this in the diagram below (the new pieces dropped at this step are in gray).



Since the three pieces we just dropped span every column, one of latter two pieces is guaranteed to be our highest piece on the board at this point. Without loss of generality, we assume the piece on the right is now our highest piece (as in the diagram above). We repeat this pattern. When we drop the horizontal piece occupying columns 3, 4, and 5, it will be stopped by the horizontal piece occupying columns 5, 6, and 7 that we dropped previously. Each of our successive pieces will then be stopped by this piece. Thus, our board looks like this:



At this point, Fames can simply drop one vertical piece in the middle and exclaim "I have earned at least one more point," as desired.



Note: If Fames repeatedly dropped vertical pieces in each column, he would eventually earn a point, but he cannot know when that has happened.

3/3/27. For $n > 1$, let a_n be the number of zeroes that $n!$ ends with when written in base n. Find the maximum value of $\frac{a_n}{n}$.

Solution

We claim that the maximum value of $\frac{a_n}{n}$ is $\frac{1}{2}$. In binary, 2! is written as 10, which ends with 1 zero. So $a_2 = 1$, and $\frac{a_2}{2} = \frac{1}{2}$. So to conclude, it suffices to show that $\frac{a_n}{n}$ is at most $\frac{1}{2}$ for all n.

Notice that the number of zeroes that $n!$ ends with when written in base n is the same as the maximum power of n that divides $n!$.

Fix n and write $n = p^k m$, where p is some prime dividing n and m is relatively prime to p. Then a_n is at most the largest power of p^k dividing $n!$. By Legendre's formula, the maximum power of p dividing $n!$ is

$$\nu_p(n!) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor .$$

So we have

$$a_n \le \frac{1}{k} \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor < \frac{1}{k} \sum_{j=1}^{\infty} \frac{n}{p^j} = \frac{n}{k} \sum_{j=1}^{\infty} \frac{1}{p^j}.$$

The sum on the far right is geometric, and sums to $\frac{1}{p-1}$. Thus,

$$a_n < \frac{n}{k(p-1)}.$$

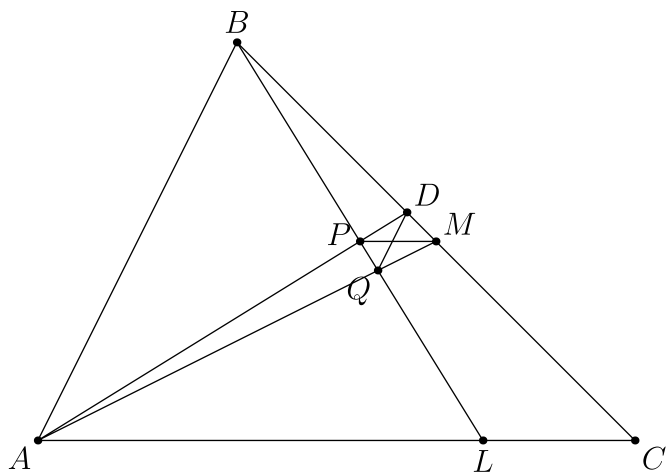
Dividing through by n gives

$$\frac{a_n}{n} < \frac{1}{k(p-1)}.$$

Therefore, if n has any prime divisor greater than 2, or is divisible by the square of any prime, $\frac{a_n}{n} < \frac{1}{2}$. So we see that the maximum value of $\frac{a_n}{n}$ is $\boxed{\frac{1}{2}}$.

4/3/27. Let $\triangle ABC$ be a triangle with $AB < AC$. Let the angle bisector of $\angle BAC$ meet BC at D, and let M be the midpoint of $\overline{BC}$. Let P be the foot of the perpendicular from B to $\overline{AD}$. Extend $\overline{BP}$ to meet $\overline{AM}$ at Q. Show that $\overline{DQ}$ is parallel to $\overline{AB}$.

Solution



Let BP meet AC at L. By construction, $\angle APB = \angle APL = 90^\circ$ and $\angle PAL = \angle PAB$ since AP is the angle bisector from A. So APB and APL are similar. Since they share side AP, we see that APB is congruent to APL. So P is the midpoint of $\overline{BL}$. Since M is the midpoint of $\overline{BC}$, this implies that BPM is similar to BLC with ratio 2, and PM is parallel to AC.

Since PM is parallel to AC, triangles DPM and DAC are similar. Hence, $\frac{MP}{AC} = \frac{DM}{DC}$. To finish, it suffices to show MDQ is similar to MBA, which means that it suffices to show that $\frac{MD}{DB} = \frac{MQ}{QA}$.

Since $\angle AQL = \angle PQM$ and PM is parallel to AL, we see that QPM is similar to QLA. Thus $\frac{MQ}{QA} = \frac{MP}{AL}$ and we just need to show that $\frac{MD}{DB} = \frac{MP}{AL}$.

Notice that if $\frac{a}{b} = \frac{c}{d} \neq \frac{1}{2}$, then $\frac{a}{b-2a} = \frac{c}{d-2c}$. Applying this to $\frac{MP}{AC} = \frac{DM}{DC}$ gives us

$$\frac{MP}{AC - 2PM} = \frac{MD}{CD - 2MD}.$$

But $2PM = LC$ because $BPM \sim BLC$ with ratio 2. So

$$\frac{MP}{AC - 2PM} = \frac{MP}{AC - LC} = \frac{MP}{AL}.$$

Similarly, we have $CM = BD + DM$, so $CD = BD + DM + DM = BD + 2DM$. Thus,

$$\frac{MD}{CD - 2MD} = \frac{MD}{BD}.$$

Putting these together, we have

$$\frac{MP}{AL} = \frac{MD}{BD},$$

as desired.

Note: This problem and solution were proposed by Xinke Guo-Xue.

5/3/27. Let $a_1, a_2, \ldots, a_{100}$ be a sequence of integers. Initially, $a_1 = 1, a_2 = -1$ and the remaining numbers are 0. After every second, we perform the following process on the sequence: for $i = 1, 2, \ldots, 99$, replace a_i with $a_i + a_{i+1}$, and replace a_{100} with $a_{100} + a_1$. (All of this is done simultaneously, so each new term is the sum of two terms of the sequence from before any replacements.) Show that for any integer M, there is some index i and some time t for which $|a_i| > M$ at time t.

Solution

Fix a positive integer M. Throughout this solution we consider indices modulo 100, so a_{102} is really just a_2. Let $a_{n,k}$ denote a_n after k seconds, with a_n being the original sequence ($a_n = a_{n,0}$). We want to show that the for some n and k we have $|a_{n,k}| > M$. We have

$$\begin{aligned} a_{1,1} &= a_1 + a_2 \\ a_{1,2} &= a_{1,1} + a_{2,1} = (a_1 + a_2) + (a_2 + a_3) = a_1 + 2a_2 + a_3 \\ a_{1,3} &= a_{1,2} + a_{2,2} = (a_1 + 2a_2 + a_3) + (a_2 + 2a_3 + a_4) = a_1 + 3a_2 + 3a_3 + a_4. \end{aligned}$$

Continuing this pattern, we claim that

$$a_{n,k} = a_n + \binom{k}{1} a_{n+1} + \binom{k}{2} a_{n+2} + \cdots$$

for all n. We show this by induction on k. It's clearly true for $k = 0$. Now suppose it's true for k and we'll show it's true for $k + 1$. By induction, we have

$$\begin{aligned} a_{n,k} &= a_{n,k-1} + a_{n+1,k-1} \\ &= \left(a_n + \binom{k-1}{1} a_{n+1} + \binom{k-1}{2} a_{n+2} + \cdots\right) + \left(a_{n+1} + \binom{k-1}{1} a_{n+2} + \binom{k-1}{2} a_{n+3} + \cdots\right) \\ &= a_n + \left(\binom{k-1}{1} + 1\right) a_{n+1} + \left(\binom{k-1}{2} + \binom{k-1}{1}\right) a_{n+2} + \cdots. \end{aligned}$$

But by Pascals identity, $\binom{k-1}{r} + \binom{k-1}{r-1} = \binom{k}{r}$. Hence, we have the result.

Of course, most of these terms are actually 0. Crossing out the 0's we have

$$\begin{aligned} a_{i,n} &= a_i + \left(\binom{n}{101-i} a_{101} + \binom{n}{201-i} a_{201} + \cdots\right) + \left(\binom{n}{102-i} a_{102} + \binom{n}{202-i} a_{202} + \cdots\right) \\ &= \sum_{k \equiv 1-i \pmod{100}} \binom{n}{k} - \sum_{k \equiv 2-i \pmod{100}} \binom{n}{k}. \end{aligned}$$

Where the last equality follows because $a_1 = 1$ and $a_2 = -1$.

Let ω be a 100th root of unity. Suppose toward a contradiction that for all i and n, $|a_{i,n}| < M$. Then by the triangle inequality

$$\left|\sum_{i=1}^{100} a_{i,n} \omega^i\right| < 100M.$$

However, we can rewrite this sum as

$$\begin{aligned}\sum_{i=1}^{100} a_{i,n}\omega^i &= \sum_{i=1}^{100}\left(\sum_{k\equiv 1-i \pmod{100}}\binom{n}{k}\omega^i - \sum_{k\equiv 2-i \pmod{100}}\binom{n}{k}\omega^i\right)\\ &= \omega\left(\sum_{k=1}^{n}\binom{n}{k}\omega^{-k} - \sum_{k=1}^{n}\binom{n}{k}\omega^{-k+1}\right).\end{aligned}$$

By the binomial theorem, we can rewrite this as

$$\omega\left((1+\omega^{-1})^n - \omega(1+\omega^{-1})^n\right) = (\omega-\omega^2)(1+\omega^{-1})^n.$$

Taking magnitudes, we have

$$\left|\sum_{i=1}^{100} a_{i,n}\omega^i\right| = \left|\omega-\omega^2\right|\cdot\left|1+\omega^{-1}\right|^n.$$

But $|1+\omega^{-1}| > 1$, so we can make this absolute value arbitrarily large by considering large enough n. This contradicts

$$\left|\sum_{i=1}^{100} a_{i,n}\omega^i\right| < 100M,$$

and we see that we must have some time t and index i for which $|a_i| > M$ at time t.

Note: This problem can be generalized. It's false if 100 is replaced by 2 or 3, but true for any other replacement, by the same argument as above.