USA Mathematical Talent Search

Solutions to Problem 2/1/19

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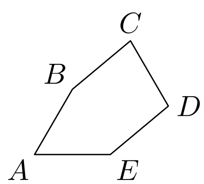
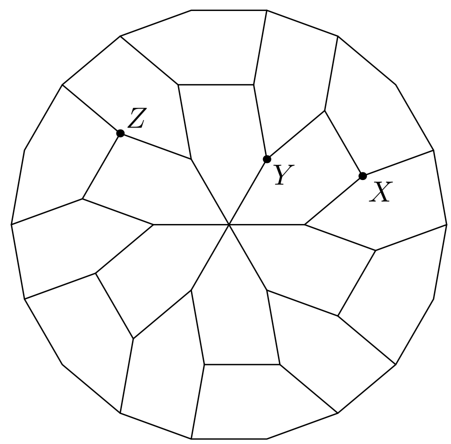
2/1/19. A regular 18-gon is dissected into 18 pentagons, each of which is congruent to pentagon $ABCDE$, as shown. All sides of the pentagon have the same length.





(a) Determine angles A, B, C, D, and E.

(b) Show that points X, Y, and Z are collinear.

Credit This problem was proposed by Naoki Sato.

Comments Part (a) can be done by considering appropriate combinations of angles in the regular 18-gon. Part (b) can be done by showing that $\angle XYZ = 180^\circ$. *Solutions edited by Naoki Sato.*

Solution 1 by: Luyi Zhang (9/CT)

(a) At the center of the 18-gon, six pentagons join together by their angle that corresponds to $\angle A$. Therefore, $\angle A = 360^\circ/6 = 60^\circ$. Since all sides of the pentagon are equal, triangle ABE is equilateral and quadrilateral $BCDE$ is a rhombus.

$\angle ABC$ is an interior angle of the 18-gon, so $\angle B = \angle ABC = 160^\circ$. Then

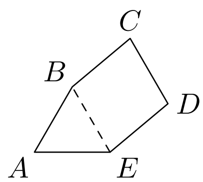
$$\angle EBC = \angle ABC - \angle ABE = 160^\circ - 60^\circ = 100^\circ,$$

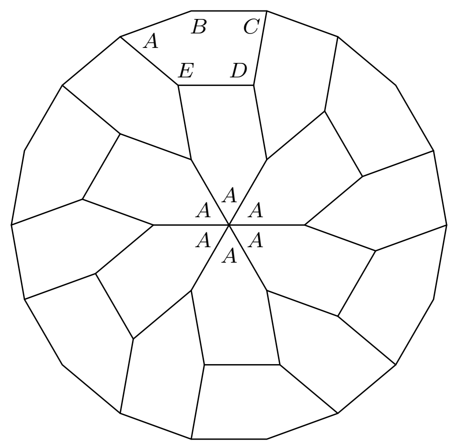
so $\angle D = \angle CDE = \angle EBC = 100^\circ$ and

$$\angle C = \angle BED = 180^\circ - \angle EBC = 180^\circ - 100^\circ = 80^\circ.$$

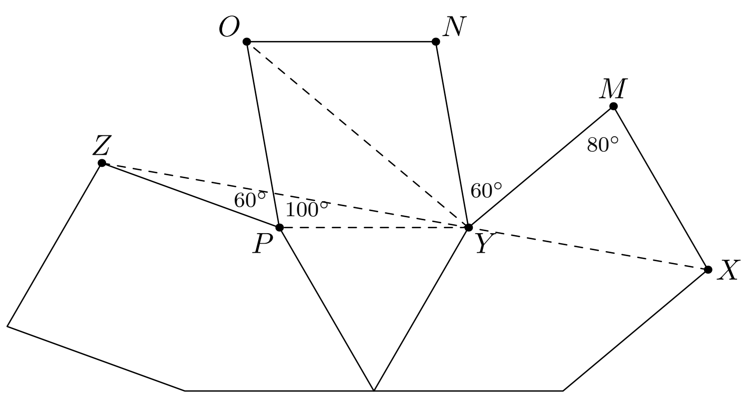
Finally, $\angle E = \angle AED = \angle AEB + \angle BED = 60^\circ + 80^\circ = 140^\circ$.

To summarize, $\angle A = 60^\circ$, $\angle B = 160^\circ$, $\angle C = 80^\circ$, $\angle D = 100^\circ$, and $\angle E = 140^\circ$.





(b) To show that points X, Y, and Z are collinear we will show that $\angle XYZ = 180^\circ$. Label points M, N, O, and P, as shown below.



Since all the sides are of equal length, we can easily create isosceles triangles to assist in our angle search. In triangle MXY, $MX = MY$ and $\angle XMY = 80^\circ$, so $\angle MXY = \angle MYX = (180^\circ - 80^\circ)/2 = 50^\circ$.

In triangle PYZ, $PY = PZ$ and $\angle ZPY = \angle ZPO + \angle OPY = 60^\circ + 100^\circ = 160^\circ$, so $\angle PZY = \angle PYZ = (180^\circ - 160^\circ)/2 = 10^\circ$.

Then in triangle OPY, $PO = PY$ and $\angle OPY = 100^\circ$, so $\angle PYO = \angle POY = \angle NYO = \angle NOY = (180^\circ - 100^\circ)/2 = 40^\circ$, so $\angle ZYO = \angle PYO - \angle PYZ = 40^\circ - 10^\circ = 30^\circ$. Then

$$\angle XYZ = \angle MYX + \angle MYN + \angle NYO + \angle ZYO = 50^\circ + 60^\circ + 40^\circ + 30^\circ = 180^\circ,$$

and we are done.