USA Mathematical Talent Search

Round 3 Grading Rubric

Year 33 — Academic Year 2021–2022

www.usamts.org

GENERAL GUIDELINES

1. The grading rubric is designed to be simultaneously **specific** and **flexible**. For common solution methods, the rubric provides a specific allocation of points to ensure consistency across graders. Less common solution methods might not be captured closely by the rubric. For less common solution methods, consider the amount of constructive progress (including any specific intermediate results discussed in the rubric) and how far or close the student is to a complete solution when determining the score.

2. On **all** problems, the graders have the discretion to deduct one additional point for a solution that is poorly written and/or hard to follow.

3. Appropriate credit should be awarded for full and partial solutions that use other correct approaches to the problem. Any solutions relying on computer methods should include the source code or specify the function call(s) (with arguments) used in a computer algebra system. Merely citing the name of a software package is not sufficient justification.

4. A student's justification needs to be rigorous and reasonably clear in order for the solution to earn **5 points**. If there is a meaningful gap in the student's argument or a key step is unclear, deduct points accordingly.

Problem 1/3/33:

Award **5 points** for the correct configuration of line segments between adjacent points. No justification is required. Withhold **1 point** for each missing line segment or each extraneous line segment.

Problem 2/3/33:

1 point: Student recognizes that thinking about parity (i.e., even vs. odd) is useful here.

4 points: Student shows that it is impossible to go from an even lattice point to an odd lattice point. Award **1 point** for finding the equation of the perpendicular bisector, **1 point** for manipulating the equation into a more useful form, **1 point** for explaining why the perpendicular bisector doesn't contain any lattice points, and **1 point** for putting all the pieces together and explaining why this means Sydney cannot reach $(2021, 2022)$.

Note: Because this is a yes/no question, don't award any credit for the correct answer unless it is accompanied by explanation.

Note: If the student proves that Sydney the Squirrel cannot move from a point with coordinates of the same parity to either a point of the form (even, odd) or to a point of the form (odd, even), but not both, award **4 points**.

Note: If the student doesn't recognize that we can go from (even, even) points to (odd, odd) points, for example $(0,0)$ to $(1,1)$, award at most **4 points**. In some cases, a lower score will be appropriate.

Note: If the student shows that it is impossible to go directly from $(0,0)$ to $(2021, 2022)$, but makes no additional progress, award **2 points**.

Problem 3/3/33:

3 points: Student proves that any pair H and V of mountains gives exactly one double staircase. Award **1 point** for recognizing, with proof, that H and V must be mountains. Award **1 point** of additional credit for significant additional constructive progress towards the result that each mountain gives exactly one double staircase, such as recognizing that for a pair of mountains H and V, T consists of the squares whose row and column sizes sum to at least $n+1$.

1 point: Student shows that in an $n \times n$ grid, there are $x_n = 2^{n-1}$ mountains.

1 point: Student recognizes that the number of double staircases is $x_n^2 = 4^{n-1} = 2^{2n-2}$.

Note: Award **1 point** for the correct answer of 4^{n-1} (or 2^{2n-2}) with no explanation.

Note: Award **2 points** for getting the answer 4^{n-1} as the number of pairs of mountains, without any proof that H and V must be mountains or that every pair of mountains gives a double staircase.

Note: Deduct **1 point** for an otherwise correct solution which says that no three consecutive rows can have i, j, k cells with $i > j < k$ because that would create a gap. The gap is only guaranteed if i or k is equal to n.

Note: Deduct **1 point** for a correct construction which does not prove that every configuration is a valid double staircase.

Note: Award **3 or 4 points** for an argument which describes how to construct the double staircases (the row of length k can be on either side of the rows longer than k) but not proving that everything works. In addition to the last **2 points**, this gets at least **1 point** for proving that H and V must be mountains, and may get another point for showing something about constructing the solutions, such as proving that there is either at least or at most one solution for each H and V.

Note: Award **1 point** for a solution giving the value $(n!)^2$, which allows every permutation of the row and column lengths (creating gaps) but correctly proves that there is one way to construct a grid with these permutations, such as by including only those cells in which the row and column lengths sum to at least $n+1$.

Note: A similar rubric applies to students using inductive arguments. Students with complete and correct solutions that use induction should receive **5 points**.

Problem 4/3/33:

Note: Be vigilant about whether the assumptions a student makes about the configuration of the diagram (including WLOG assumptions) are valid. This is a place where it is easy to make a mistake.

Note: If the student used a method that was significantly different from the official solution, we tended to grade the solution holistically, keeping in mind the breakdown below to determine the quality of the student's constructive progress.

2 points: Student shows for a significant special case (e.g., the circumcircle of ABC is completely contained within Ω) that we can choose two of the vertices of ABC such that there are infinitely many circles ω that satisfy the three conditions in this problem. Award **1 point** for significant constructive progress towards this result, such as recognizing that it is useful to think about open segment MN, where M and N are the midpoints of segment and minor arc AB respectively.

3 points: Student extends their reasoning from the special case to address all other cases. Award partial credit as appropriate for significant constructive progress, such as addressing additional special cases and/or doing useful transformations such as the homothety in the official solution.

Problem 5/3/33:

Note: Award **3 points** for successfully proving one direction, and **5 points** for successfully proving both directions.

1 point: Student uses the property of the floor function to obtain a meaningful result, such as recognizing that $\frac{\left\lfloor \frac{x+a}{b} \right\rfloor + c}{d}$ and $\frac{x+e}{f}$ must differ by less than 1.

2 points: Student proves that if there exist e, f that satisfy the equation, then d is an integer. Award **1 point** of partial credit for significant constructive progress towards this result, such as recognizing that $f = bd$, or an equivalently meaningful result.

2 points: Student proves that if d is an integer, then there exist e, f that satisfy the equation. Award **1 point** of partial credit for significant constructive progress towards this result, such as proving the Lemma and applying it in a meaningful way to the problem.

Note: Award **1 point** for showing that b is redundant: we can replace a, b, c, d, e, f by $a/b, 1, c, d, e/b, f/b$. This is significant progress because it simplifies the condition we need to $d = f$.

Note: Award **2 points** for proving $bd = f$ formally using growth rates (the left side grows by an average of 1 when x is increased by bd, and the right side grows by 1 when x is increased by f, so if $bd < f$, or $bd > f$, the left side is more/less than the right side for large enough x). Deduct **1 point** for an informal argument, and thus award a maximum of **1 of the 2 points** for proving that d is an integer.

Note: A common error is the solution $e = a + bc$ (rather than $e = a + b\lfloor c \rfloor$). This is usually the result of making some error with the floor function in an otherwise correct argument, and thus leads to a **1-point** deduction, but sometimes there is a more fundamental error, or it affects the proof of both halves, and **2 points** must be deducted.

Note: Do not deduct a point for failing to prove that the solution for d an integer gives positive e and f if a formula for e and f is given, as it is clear that they are positive. Award only **4 points** for an otherwise correct solution with no formula for e, so that it is not clear that e is positive.

Note: Do not award any points for only proving that e and f must be positive (since the left-hand side is non-negative for $x \geq -a$).