USA Mathematical Talent Search

Round 3 Grading Rubric

Year 32 — Academic Year 2020–2021

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GENERAL GUIDELINES

1. The grading rubric is designed to be simultaneously **specific** and **flexible**. For common solution methods, the rubric provides a specific allocation of points to ensure consistency across graders. Less common solution methods might not be captured closely by the rubric. For less common solution methods, consider the amount of constructive progress (including any specific intermediate results discussed in the rubric) and how far or close the student is to a complete solution when determining the score.

2. On **all** problems, the graders have the discretion to deduct one additional point for a solution that is poorly written and/or hard to follow.

3. Appropriate credit should be awarded for full and partial solutions that use other correct approaches to the problem. Any solutions relying on computer methods should include the source code or specify the function call(s) (with arguments) used in a computer algebra system. Merely citing the name of a software package is not sufficient justification.

4. A student's justification needs to be rigorous and reasonably clear in order for the solution to earn **5 points**. If there is a meaningful gap in the student's argument or a key step is unclear, deduct points accordingly.

Problem 1/3/32:

Award **5 points** for the correct configuration of numbers. No justification is required. Withhold **1 point** for each incorrect entry.

Note: Adjacent entries need to have either the same tens digit or the same ones digit. For example, 31 and 17 cannot be adjacent, since 1 is the ones digit of 31 but the tens digit of 17. This was a common error.

Problem 2/3/32:

Note: A correct answer, including stating the distances, is worth **2 points**, and the explanation is worth **3 points**.

2 points: Student states the coordinates of a valid set of four points (**1 point**) and correctly states the six distances (**1 point**).

3 points: Student gives a reasonable explanation of **how they obtained their configuration of points**. In other words, we are awarding points here for the student's explanation of their strategy, not for using the distance formula to verify that their configuration works. Most students gave reasonably detailed explanations, but there were some limits with respect to awarding full credit. In cases where the student's explanation was insufficient, we tried to explain what we thought was lacking.

Note: If a student has a length other than AB be the even length but has the right number of odd and even lengths, give full credit but comment accordingly.

Note: If a student's configuration was incorrect, the most likely score was **0 points**. If a student attempted to use cyclic quadrilaterals in a meaningful way, used a triangle with a $120°$ angle and pointed out that $\cos 120°$ is rational, or made significant constructive progress using another potentially viable strategy, we typically gave a total score of **1 point**. We also gave a total score of **1 point** if the student explained rigorously why a configuration based on Pythagorean triples can't give us the right number of even and odd lengths, but didn't make additional constructive progress.

Problem 3/3/32:

2 points: Student shows that all squarefree integers n satisfy the conditions in the problem. Award **1 point** of partial credit for significant constructive progress towards this result, such as a meaningful application of the formula for the number of divisors of a positive integer m.

3 points: Student shows that all non-squarefree integers n (except for 4) do not satisfy the conditions in the problem. The case in which p is odd is worth **2 points** and the case in which $p = 2$ is worth **1 point**. For the case in which p is odd, a student should receive **1 point** of partial credit for recognizing that

$$m_q = p^{p^{e-1}-1} \cdot p_1^{p_1^{e_1}-1} p_2^{p_2^{e_2}-1} \cdots p_k^{p_k^{e_k}-1} \cdot q^{p-1}$$

is a positive multiple of n with exactly n positive divisors.

Note: Award **1 point** for the correct answer, with or without justification.

Note: If the student shows that $n = 1$ and $n = 4$ work, but makes no other constructive progress, award **1 point**.

Note: Award at most **4 points** if a student misses $n = 1$ and/or $n = 4$, with the following caveat. Since 1 is considered to be squarefree, it is included in the empty subcase of $n = p_1 \cdot p_2 \cdot p_3 \cdots p_k$. If this was implicit in the student's solution, we awarded **5 points**.

Problem 4/3/32:

Note: Students tended to arrive at the general inequality

$$(n-20)(n-21) \leq n(n-1) \cdot \frac{n-k}{n},$$

and then they applied the inequality to each part of the problem. Award **3 points** for obtaining the aforementioned inequality with proof, **1 point** for successfully applying the inequality to part (a), and **1 point** for successfully applying the inequality to part (b).

Note: The above allocation of points makes it hard to give more than **2 points** to a solution that arrives at an incorrect general inequality. If the student basically has the right ideas, but a small mistake leads to an incorrect general inequality, graders may award a total score up to (and including) **4 points** since applying the inequalities to each part is relatively easy.

Note: Many (but not all) students used ideas from graph theory (similar to the official solution) to obtain the general inequality. For graph theory solutions, award the **3 points** for obtaining the general inequality as follows:

1 point: Student sets up the problem using graph theory, and obtains a significant expression, such as the number of edges of all induced subgraphs given by taking a vertex subset of size $n-20$:

$$T = \sum_{S \subseteq V(G), |S|=n-20} |E(G[S])|.$$

1 point: Student uses the property that the number of pairs of people that know each other is at most $\frac{n-k}{n}$ times that of the original group of people to obtain a useful expression, such as

$$|E(G[S])| \leq \frac{n-k}{n} \cdot |E(G)|.$$

1 point: Student uses the above observations, and completes the additional steps to show that

$$(n-20)(n-21) \leq n(n-1) \cdot \frac{n-k}{n}.$$

Note: For full credit in part (b), the example graph of the complete graph, K_n, must be given to show attainability.

Note: Award at most **3 points** if the student uses the greedy algorithm to "remove the 20 vertices of lowest degree." The greedy algorithm may not be the way to remove the fewest number of edges, depending on the distribution of edges within these 20 vertices versus between these vertices and the rest of the graph.

Note: If a student assumes an equality in part (b), i.e., $n = 381$ instead of $n \geq 381$, deduct **1 point**.

Note: "By the probabilistic method" is sufficient argument for justifying the inequality.

Problem 5/3/32:

1 point: Student shows that if $n \geq 3$ is prime, then the function f given by $f(x) = x \pmod{n}$ is not of the form $f(x) = cx + d$, but has the desired property.

2 points: Student shows that if $f(m) = f(m + n) = k$, then f must be a constant function. Award **1 point** of partial credit for significant constructive progress towards this result, such as showing that

$$f(m), f(m + 1), f(m + 2), \cdots , f(m + n - 1)$$

must be a constant arithmetic sequence, and attempting to use this result in a meaningful way.

2 points: Student shows that if $f(m) \neq f(m + n)$ for all m, then f must be linear. Award **1 point** of partial credit for significant constructive progress towards this result, such as showing that $f(m)$ must either be the first or last term of the arithmetic sequence given by $f(m), f(m + 1), \cdots , f(m + n - 1)$, (i.e., either $f(m) = a$ or $f(m) = a + (n - 1)d$), and attempting to use this result in a meaningful way.