USA Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS
Round 1 - Year 13 - Academic Year 2001-2002

Gene A. Berg, Editor

1/1/13. Determine the unique positive two-digit integers m and n for which the approximation $\frac{m}{n} \approx 0.2328767$ is accurate to seven decimals, i.e., $0.2328767 \le \frac{m}{n} < 0.2328768$.

Solution 1 by Grigori Avramidi (11/NM): We can use Euclid's Algorithm to obtain a continued fraction approximation for 0.2328767 and then check whether it fits our bounds. A continued fraction is of the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\dots}}}}.$$

It can also be written as $[a_1, a_2, a_3, a_4, \dots]$. This version of Euclid's Algorithm works as follows. One takes a number and expresses it as the greatest integer of it (this is the number in parenthesis) plus the remainder. This greatest integer corresponds to a_1 in the continued fraction. Next one takes the inverse of the remainder and does the same thing. This can be repeated an unlimited number of times, each reiteration increasing the accuracy of the calculation. A simple continued fraction may be finite or infinite. It may also be repeating.

$$0.2328767 = (0) \cdot 1 + 0.2328767 \tag{1}$$

$$\frac{1}{0.2328767} \approx (4) \cdot 1 + 0.2941179 \tag{2}$$

$$\frac{1}{0.2941179} \approx (3) \cdot 1 + 0.3999974 \tag{3}$$

$$\frac{1}{0.3999974} \approx (2) \cdot 1 + 0.5000164 \tag{4}$$

$$\frac{1}{0.5000164} \approx (1) \cdot 1 + 0.9999344 \tag{5}$$

$$\frac{1}{0.9999344} \approx (1) \cdot 1 + 0.0000656 \tag{6}$$

$$\frac{1}{0.0000656} \approx (15243) \cdot 1 + 0.9024390 \tag{7}$$

This tells us that 0.2328767 is approximated by the finite continued fraction

$[0, 4, 3, 2, 1, 1, 15234]$.

Now we set up a table to obtain the convergents to 0.2328767.

$m_x = a_x(m_{x-1} + m_{x-2})$

$n_x = a_x(n_{x-1} + n_{x-2})$

x	-	-	1	2	3	4	5	6	7
a_x	-	-	0	4	3	2	1	1	15243
m_x	0	1	0	1	3	7	10	17	259141
n_x	1	0	1	4	13	30	43	73	1112782

We see that the closest approximation $\frac{m}{n}$ where m and n are two-digit numbers is $\frac{17}{73}$. We check and find that $\frac{17}{73} = 0.23287671\ldots$ so $0.2328767 \le \frac{17}{73} < 0.2328768$.

Thus

$$m = 17 \text{ and } n = 73.$$

Another way to evaluate a continued fraction is by writing out the continued fraction.

$$0 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1}}}}} = 0 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{(5/2)}}} = 0 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{2}{5}}} = \cfrac{1}{4 + \cfrac{5}{17}} = \frac{17}{73}.$$

We see that we get the same result

$$m = 17 \text{ and } n = 73.$$

Solution 2 by Mike Hansen (12/IA): We will find m and n by the continued fractions method.

$$0.2328767 \approx \frac{1}{4 + 0.29411787} \approx \cfrac{1}{4 + \cfrac{1}{3 + 0.3999974}} \approx \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{2}}}}$$

$$\approx \cfrac{1}{4 + \cfrac{1}{3 + 0.39999737}}$$

$$\approx \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{2 + 0.50001643}}}$$

$$\approx \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{2 + \frac{1}{2}}}}$$

$$= \frac{17}{73} = \frac{m}{n} \approx 0.2328767$$

Notice that we "stopped" at $\frac{1}{2}$ because if we kept going, *m* and *n* would not both be two digit numbers.

So $m = 17$ and $n = 73$.

Solution 3 by Shyam Amin (10/NY): Use Farey sequences with $\alpha = 0.2328767$.

$0 = \frac{0}{1} < \alpha < \frac{1}{1} = 1$. The next fraction between $\frac{0}{1}$ and $\frac{1}{1}$ with least denominator is $\frac{1}{2} = 0.5$.

$\frac{0}{1} < \alpha < \frac{1}{2}$. The next fraction between $\frac{0}{1}$ and $\frac{1}{2}$ with least denominator is $\frac{1}{3} = 0.\bar{3}$.

$\frac{0}{1} < \alpha < \frac{1}{3}$. The next fraction between $\frac{0}{1}$ and $\frac{1}{3}$ with least denominator is $\frac{1}{4} = 0.25$.

$\frac{0}{1} < \alpha < \frac{1}{4}$. The next fraction between $\frac{0}{1}$ and $\frac{1}{4}$ with least denominator is $\frac{1}{5} = 0.20$.

$\frac{1}{5} < \alpha < \frac{1}{4}$. The next fraction between $\frac{1}{5}$ and $\frac{1}{4}$ with least denominator is $\frac{2}{9} = 0.\bar{2}$.

$\frac{2}{9} < \alpha < \frac{1}{4}$. The next fraction between $\frac{2}{9}$ and $\frac{1}{4}$ with least denominator is $\frac{3}{13} = 0.2307692\ldots$.

$\frac{3}{13} < \alpha < \frac{1}{4}$. The next fraction between $\frac{3}{13}$ and $\frac{1}{4}$ with least denominator is $\frac{4}{17} = 0.2352941\ldots$.

$\frac{3}{13} < \alpha < \frac{4}{17}$. The next fraction between $\frac{3}{13}$ and $\frac{4}{17}$ with least denominator is $\frac{7}{30} = 0.2\bar{3}$.

$\frac{3}{13} < \alpha < \frac{7}{30}$. The next fraction between $\frac{3}{13}$ and $\frac{7}{30}$ with l. d. is $\frac{10}{43} = 0.232558\ldots$.
$\frac{10}{43} < \alpha < \frac{7}{30}$. The next fraction between $\frac{10}{43}$ and $\frac{7}{30}$ with l. d. is $\frac{17}{73} = 0.2328767\ldots$.

When $m = 17$ and $n = 73$, then $\frac{m}{n} \approx 0.2328767$.

Editor's Comment: This problem, created by our Problem Editor, George Berzsenyi, was inspired by a similar problem in the January 1999 issue of *Mathematical Digest*. It is also related to problem 2/3/12; *continued fractions* and *Farey sequences* are discussed further in the solutions to that problem available on our web site.

2/1/13. It is well known that there are infinitely many triples of integers (a, b, c) whose greatest common divisor is 1 and which satisfy the equation $a^2 + b^2 = c^2$. Prove that there are also infinitely many triples of integers (r, s, t) whose greatest common divisor is 1 and which satisfy the equation $(rs)^2 + (st)^2 = (rt)^2$.

Solution by Anna Pierrehumbert (11/IL):

Claim: If *a*, *b*, and *c* are integers such that $a^2 + b^2 = c^2$ and $\gcd(a, b, c) = 1$, then $r = ac$, $s = ab$, and $t = bc$ satisfy $(rs)^2 + (st)^2 = (rt)^2$ and $\gcd(r, s, t) = 1$.

Proof: Multiplying $a^2 + b^2 = c^2$ by $(abc)^2$ gives $(a^2bc)^2 + (ab^2c)^2 = (abc^2)^2$, which can be written as $((ac)(ab))^2 + ((ab)(bc))^2 = ((ac)(bc))^2$. Letting $r = ac$, $s = ab$, and $t = bc$. yields $(rs)^2 + (st)^2 = (rt)^2$. It is given that $\gcd(a, b, c) = 1$. Furthermore, no two of *a*, *b*, and *c* have a common factor, because if they did, the common factor would be factored from two terms of $a^2 + b^2 = c^2$ and not the third. Therefore, because *r*, *s*, and *t* share no common factor *a*, *b*, or *c*, $\gcd(r, s, t) = 1$.

There are infinitely many integers a, b, c such that $a^2 + b^2 = c^2$ and $\gcd(a, b, c) = 1$. As shown above, each triple (a, b, c) corresponds to a triple (r, s, t) satisfying $(rs)^2 + (st)^2 = (rt)^2$ and $\gcd(r, s, t) = 1$. Consequently, there are infinitely many triples (r, s, t) satisfying $(rs)^2 + (st)^2 = (rt)^2$ and $\gcd(r, s, t) = 1$. Q.E.D.

Editor's Comment: The author of this problem is Mr. Surat Intasang, a former USAMTS participant, who recently initiated the equivalent of the USAMTS in his native Thailand. We thank him for his contribution and wish him well in his endeavors.

3/1/13. Suppose $\frac{\cos 3x}{\cos x} = \frac{1}{3}$ for some angle *x*, $0 \le x \le \frac{\pi}{2}$. Determine $\frac{\sin 3x}{\sin x}$ for the same *x*.

Solution 1 by Julia Mundy (12/NY): There are two methods of solving this problem. The use of technology will produce an explicit value for x. This value can be substituted into the second expression and the answer found. However, the solution can also be found more elegantly without the use of technology or an explicit value for x.

$\cos(3x) = \cos(2x + x)$. This expression can be evaluated using the trigonometric double angle and summation rules.

$$\cos(3x) = \cos(2x)\cos(x) - \sin(2x)\sin(x)$$
$$= \frac{(1 - 2\sin^2 x)\cos(x) - 2\sin(x)\cos(x)}{\sin(x)}$$

Factoring out $\cos(x)$ and simplifying, we receive:

$$\cos(3x) = \cos(x)(-4\sin^2 x + 1) \quad .$$

Therefore, $\frac{\cos(3x)}{\cos(x)} = -4\sin^2 x + 1 = \frac{1}{3}$.

We can find a similar expression for $\frac{\sin(3x)}{\sin(x)}$ using the trigonometric identities:

$$\sin(3x) = \sin(x)(4\cos^2 x - 1) \quad \text{and} \quad \frac{\sin(3x)}{\sin(x)} = 4\cos^2 x - 1 \quad .$$

$$\frac{\cos(3x)}{\cos(x)} = -4\sin^2 x + 1 = \frac{1}{3} \quad .$$

As $\sin^2 x = 1 - \cos^2 x$, we can substitute to receive

$$\frac{1}{3} = -4(1 - \cos^2 x) + 1$$
$$\frac{1}{3} = -4\cos^2 x - 3$$

or $\frac{7}{3} = -4\cos^2 x - 1$, which is our expression for $\frac{\sin(3x)}{\sin(x)}$.

So, $\frac{\sin(3x)}{\sin(x)} = \frac{7}{3}$.

Solution 2 by Victor Li (11/CA):

$$\begin{aligned}\frac{\sin(3x)}{\sin(x)} - \frac{\cos(3x)}{\cos(x)} &= \frac{\sin(3x)\cos(x) - \cos(3x)\sin(x)}{\sin(x)\cos(x)} \\ &= \frac{\sin(2x)}{\sin(x)\cos(x)} \\ &= 2 \cdot \frac{\sin(2x)}{\sin(2x)} \\ &= 2\,.\end{aligned}$$

So,

$$\frac{\sin(3x)}{\sin(x)} - \frac{1}{3} = 2$$

or

$$\frac{\sin(3x)}{\sin(x)} = 2 + \frac{1}{3} = \frac{7}{3}$$

Editor's Comment: We are indebted to Dr. Bruce Reznik of the University of Illinois for the idea leading to this problem. By the way, the value of x is near, but not equal to $24°$.

4/1/13. The *projective plane* of order three consists of 13 "points" and 13 "lines". These lines are not Euclidean straight lines; instead they are sets of four points with the properties that each pair of lines has exactly one point in common, and each pair of points has exactly one line that contains both points. Suppose the points are labeled 1 through 13, and six of the lines are $A = \{1, 2, 4, 8\}$, $B = \{1, 3, 5, 9\}$, $C = \{2, 3, 6, 10\}$, $D = \{4, 5, 10, 11\}$, $E = \{4, 6, 9, 12\}$, and $F = \{5, 6, 8, 13\}$. What is the line that contains 7 and 8?

Solution 1 by Bob Hough (11/MI): Observe that if there are 13 lines, 13 points, 4 points per line where no point can occur in a given line more than once, then each point must appear in 4 distinct lines. Since no point may appear with another point in more than one line, any pairing of two points on a given line is unique to that line. Therefore, if three of the lines containing a given point are known, then the points that constitute the fourth line containing that point are also known. There are three points that occur three times in lines A - F. They are 4, 5, and 6. Therefore, it is possible to decide that lines {3,4,7,13}, {2,5,7,12}, and {1,6,7,11} are in the projective plane. Now given that three lines are known that contain 7, it is possible to determine that the fourth line containing both 7 and 8 is {7,8,9,10}.

Solution 2 by Scott Wilbur (11/MA): Here are the lines we know:

A:	1	2	4	8
B:	1	3	5	9
C:	2	3	6	10
D:	4	5	10	11
E:	4	6	9	12
F:	5	6	8	13

Because each set of two points defines a line, and:

4 has not yet been used with 3, 7, or 13;
5 has not yet been used with 2, 7, or 12; and
6 has not yet been used with 1, 7, or 11;

we know that these lines are also a part of the plane:

G:	4	3	7	13
H:	5	2	7	12
I:	6	1	7	11

Because 7 has not yet been used with 8, 9, or 10, we can add this line:

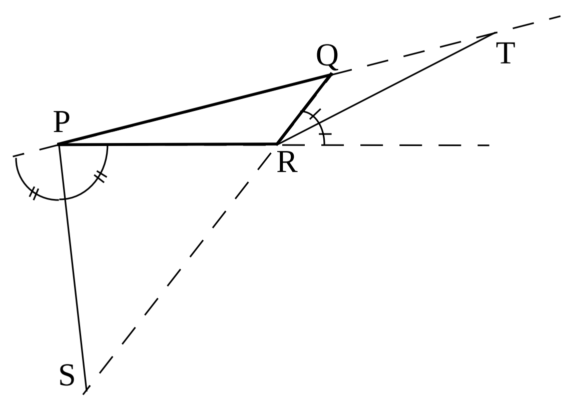
J:	7	8	9	10

This is the line we are looking for:

$$\{7, 8, 9, 10\}.$$

Editor's comment: This problem was proposed by Dr. Erin Schram of the National Security Agency. We are most thankful for his continuing support of the USAMTS.

5/1/13. In ΔPQR, $QR < PR < PQ$ so that the exterior angle bisector through P intersects ray $\overrightarrow{QR}$ at point S, and the exterior angle bisector at R intersects ray $\overrightarrow{PQ}$ at point T, as shown on the right. Given that $PR = PS = RT$, determine, with proof, the measure of $\angle PRQ$.

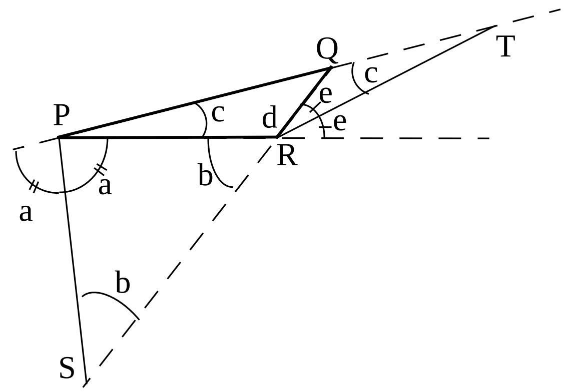


Solution 1 by Elena Ruse (10/NY):
(Note: All angles are measured in degrees.)
Let X be a point on PR with R between P and X. Let $m(\angle QRX) = \phi$. Notice that we need to find $180 - \phi$. Since $\angle TRX$ is exterior to triangle TPR, $\frac{\phi}{2} = m(\angle TRX) = 2m(\angle TPR)$, because TPR is isosceles. So $m(\angle TPR) = \frac{\phi}{4}$. Then $m(\angle RPS) = \frac{\left(180 - \frac{\phi}{4}\right)}{2} = 90 - \frac{\phi}{8}$. Now

we know all three angles of isosceles triangle PSR: $90 - \frac{\phi}{8}$, ϕ, and ϕ (because $\angle PSR = \angle PRS = \angle QRS$). So $90 - \frac{\phi}{8} + \phi + \phi = 180$; $\frac{15\phi}{8} = 90$; $\phi = 48$. So $180 - \phi = 132$.

So the measure of $\angle PRQ$ is 132 degrees.

Solution 2 by Kristin Berger (11/PA): As shown on the right, each essential angle measure is assigned a variable.



The following system of five linear equations in five unknowns is obtained:

$2a + c = 180°$	$\angle$ addition property for Δs
$2b + a = 180°$	Isosceles Δ Thm., $\angle$ addition property
$b + d = 180°$	Supplementary $\angle s$
$b - 2e = 0°$	Vertical $\angle s$ are equal
$2c + d + e = 180°$	Isosceles Δ Thm., $\angle$ addition property for Δs

This system of equations forms the following matrix equation:

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 180 \\ 180 \\ 180 \\ 0 \\ 180 \end{bmatrix}$$

Solving this matrix equation yields:

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 180 \\ 180 \\ 180 \\ 0 \\ 180 \end{bmatrix} = \begin{bmatrix} \frac{8}{13} & \frac{-3}{13} & \frac{4}{13} & \frac{2}{13} & \frac{-4}{13} \\ \frac{-4}{13} & \frac{8}{13} & \frac{-2}{13} & \frac{-1}{13} & \frac{2}{13} \\ \frac{-3}{13} & \frac{6}{13} & \frac{-8}{13} & \frac{-4}{13} & \frac{8}{13} \\ \frac{4}{13} & \frac{-8}{13} & \frac{15}{13} & \frac{1}{13} & \frac{-2}{13} \\ \frac{2}{13} & \frac{-4}{13} & \frac{1}{13} & \frac{7}{13} & \frac{-1}{13} \end{bmatrix} \begin{bmatrix} 180 \\ 180 \\ 180 \\ 0 \\ 180 \end{bmatrix} = \begin{bmatrix} 84 \\ 48 \\ 12 \\ 132 \\ 24 \end{bmatrix}$$

Each of the end values corresponds to an angle value. Notice there has been no loss of accuracy since each expression is exact.

$\angle d$ corresponds to $\angle PRQ$, so $m(\angle PRQ) = 132°$.

Editor's comments: This triangle is known as the *Bottema triangle*. To learn more about it, see I. F. Sharygin's article on "The Steiner-Lehmus Theorem" in the November-December 1998 issue of *Quantum* magazine. Unfortunately, *Quantum* is no longer being published. This problem was proposed by Dr. Berzsenyi.