1/1/30. Fill in each space of the grid with one of the numbers $1, 2, \ldots, 30$, using each number once. For $1 \le n \le 29$, the two spaces containing n and $n+1$ must be in either the same row or the same column. Some numbers have been given to you.

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

29					
	19			17	
13			21		8
	4		15		24
10				26	

Solution

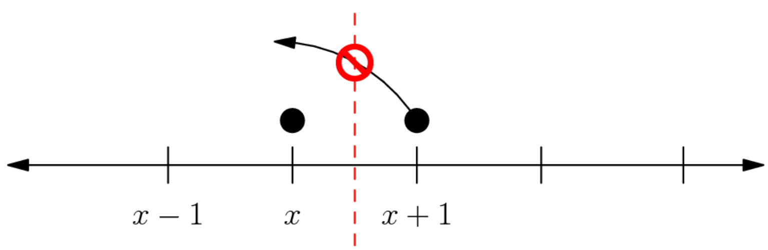
29	3	30	22	28	23
18	19	6	16	17	7
13	20	12	21	27	8
14	4	5	15	25	24
10	2	11	1	26	9

2/1/30. Let $n > 1$ be an integer. There are n orangutoads, conveniently numbered $1, 2, \ldots, n$, each sitting at an integer position on the number line. They take turns moving in the order $1, 2, \ldots, n$, and then going back to 1 to start the process over; they stop if any orangutoad is ever unable to move. To move, an orangutoad chooses another orangutoad who is at least 2 units away from her and hops towards them by a distance of 1 unit. (Multiple orangutoads can be at the same position.) Show that eventually some orangutoad will be unable to move.

Solution

At the start of some round, suppose that the leftmost orangutoad is at position x on the number line. (By "round," we mean a sequence of n moves by orangutoads 1, 2, $\ldots$, n.)

When the first orangutoad, 1, moves in this round, she cannot hop to position x or to the left of x. Indeed, to do so, she would need to start at position x or $x + 1$ and hop left, but since no orangutoads are at position $x - 1$ or to the left of $x - 1$, it is impossible for an orangutoad at position x or $x + 1$ to hop left.



Thus, orangutoad 1 must end up at position $x + 1$ or to the right of $x + 1$. The same argument applies to orangutoad 2, since when it is her turn, there will still be no orangutoads at or to the left of position $x - 1$. Therefore, we see that every orangutoad must end up at position $x + 1$ or to the right of $x + 1$ after the round.

Similarly, at the start of a given round, if the rightmost orangutoad is sitting at position y on the number line, then after the round, every orangutoad will be at a position less than or equal to $y - 1$.

Thus, after each round, the distance between the leftmost and rightmost orangutoads must decrease by at least 2. This distance can never be negative, so there can only be a finite number of rounds. That is, eventually some orangutoad must be unable to move.

3/1/30. Find, with proof, all pairs of positive integers (n, d) with the following property: for every integer S, there exists a unique non-decreasing sequence of n integers $a_1, a_2, \ldots, a_n$ such that $a_1 + a_2 + \cdots + a_n = S$ and $a_n - a_1 = d$.

Solution

We try to eliminate some possibilities for n and d. First note that, if $n = 1$, then we have $d = a_n - a_1 = a_1 - a_1 = 0$, which is impossible because we are given that $d > 0$. Therefore, all pairs (n, d) with the given property must have $n \geq 2$.

If $n = 2$, then all such sequences take the form $(a_1, a_2) = (a_1, a_1 + d)$ for some a_1 and d. The sum of such a sequence is $a_1 + (a_1 + d) = 2a_1 + d$, which is always even if d is even or always odd if d is odd, so the sum of the sequence cannot equal every integer S. Therefore, we must have $n \geq 3$.

If $d = 1$, then for any such sequence (a_k), we have

$$(n-1)a_1 + (a_1 + 1) \leq a_1 + a_2 + \cdots + a_n \leq a_1 + (n-1)(a_1 + 1),$$

or

$$na_1 + 1 \leq a_1 + a_2 + \cdots + a_n \leq na_1 + (n-1).$$

In particular, $a_1 + a_2 + \cdots + a_n$ can never be a multiple of n, so when S is a multiple of n, there is no such sequence (a_k). Therefore, all pairs (n, d) with the given property must have $d \geq 2$.

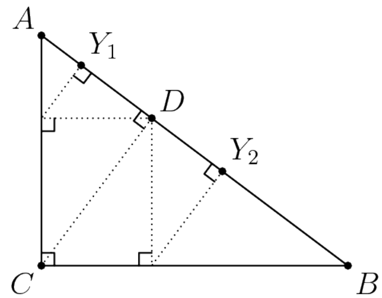
If $n \geq 4$, then for $S = 2d$, the n-term sequences $(0, 0, \ldots, 0, d, d)$ and $(0, \ldots, 0, 1, d-1, d)$, with $n - 2$ and $n - 3$ zero terms, respectively, satisfy all the conditions, so there is not a unique such sequence (a_k). Therefore, the only possibility is $n = 3$.

If $d \geq 3$, then for $S = d$, the sequences $(0, 0, d)$ and $(-1, 2, d-1)$ satisfy all the conditions, so there is not a unique such sequence (a_k). Therefore, the only possibility is $d = 2$.

We now prove that the pair $(n, d) = (3, 2)$ has the given property. For $n = 3$ and $d = 2$, all the non-decreasing sequences (a_1, a_2, a_3) with $a_3 - a_1 = 2$ take one of the forms $(t, t, t+2)$, $(t, t+1, t+2)$, or $(t, t+2, t+2)$, where t is an integer. The sum of the terms in each of these sequences is $3t + 2$, $3t + 3$, and $3t + 4$, respectively. Every integer S can be written uniquely in one of these three forms, so each S corresponds to a unique sequence (a_1, a_2, a_3).

Thus, the only pair (n, d) with the given property is $\boxed{(3, 2)}$.

4/1/30. Right triangle $\triangle ABC$ has $\angle C = 90°$. A fly is trapped inside $\triangle ABC$. It starts at point D, the foot of the altitude from C to $\overline{AB}$, and then makes a (finite) sequence of moves. In each move, it flies in a direction parallel to either $\overline{AC}$ or $\overline{BC}$; upon reaching a leg of the triangle, it then flies to a point on $\overline{AB}$ in a direction parallel to $\overline{CD}$. For example, on its first move, the fly can move to either of the points Y_1 or Y_2, as shown.



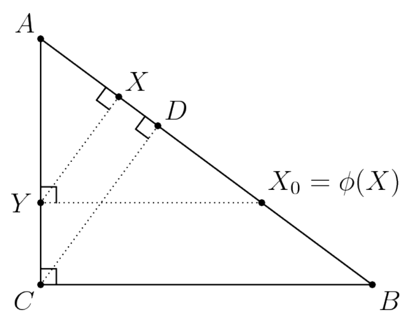
Let P and Q be distinct points on $\overline{AB}$. Show that the fly can reach some point on $\overline{PQ}$.

Solution

Note that, for any point X on open segment $\overline{AD}$, there is a unique point X_0 on open segment $\overline{AB}$ such that the fly can get from X_0 to X in one move. This is because the fly's motion is reversible: given X, we can draw the line through X perpendicular to $\overline{AB}$, mark its intersection Y with $\overline{AC}$, draw the line through Y perpendicular to $\overline{AC}$, and then mark its intersection with $\overline{AB}$ to find X_0, the previous position of the fly. The same is true for any point X on open segment $\overline{DB}$.

Let ϕ be the map, defined on the union of open segments $\overline{AD}$ and $\overline{DB}$, which assigns to each point X the fly's previous position X_0. For each X on open segment $\overline{AD}$, we have $\triangle AX_0Y \sim \triangle ABC$. Since X and D are are both the feet of altitudes from the right angles in these triangles, it follows that

$$\frac{AX_0}{AX} = \frac{AB}{AD}.$$



This means that on open segment $\overline{AD}$, we know that ϕ is just a dilation centered at A with ratio $\frac{AB}{AD} > 1$. Similarly, on open segment $\overline{DB}$, we know that ϕ is a dilation centered at B with ratio $\frac{AB}{DB} > 1$.

Now we turn to the problem at hand. It suffices to prove the statement for sufficiently small segments $\overline{PQ}$, so we may assume that P and Q lie on the same side of D and are distinct from A, B, and D. Suppose for the sake of contradiction that the fly can never reach any point on $\overline{PQ}$, starting at D. For each nonnegative integer n, consider the set $\phi^n(\overline{PQ})$, the result of n applications of ϕ to the interval $\overline{PQ}$. Since ϕ is a dilation when restricted to either open segment $\overline{AD}$ or open segment $\overline{DB}$, it follows that the image of any interval contained in $\overline{AD}$ or $\overline{DB}$ is another interval. Our assumption that the fly can never reach a point in $\overline{PQ}$ implies that D does not lie in $\phi^n(\overline{PQ})$ for every n, so by an inductive argument, $\phi^n(\overline{PQ})$ is an interval contained either in open segment $\overline{AD}$ or in open segment $\overline{DB}$. In particular, for each n, the length of the interval $\phi^n(\overline{PQ})$ is at most AB. But, letting $r = \min(\frac{AB}{AD}, \frac{AB}{DB})$, the length of the interval $\phi^n(\overline{PQ})$ is at least $r^n PQ$ for each n, which blows up as $n \to \infty$ because $r > 1$. This is a contradiction, and completes the proof.

5/1/30. A positive integer is called *uphill* if the digits in its decimal representation form a non-decreasing sequence from left to right. That is, a number with decimal representation $\overline{a_1 a_2 \ldots a_d}$ is uphill if $a_i \le a_{i+1}$ for all i. (All single-digit integers are uphill.)

Given a positive integer n, let $f(n)$ be the smallest nonnegative integer m such that $n+m$ is uphill. For example, $f(520) = 35$ and $f(169) = 0$. Find, with proof, the value of

$$f(1) - f(2) + f(3) - f(4) + \cdots + f(10^{2018} - 1).$$

Solution

First, note that $f(1) = 0$ since 1 is uphill, so we can rewrite our sum as

$$-\left(f(2) - f(3) + f(4) - f(5) + \cdots + f(10^{2018} - 2) - f(10^{2018} - 1)\right).$$

Now we can pair up consecutive terms so the sum becomes

$$-[f(2) - f(3)] - [f(4) - f(5)] - \cdots - [f(10^{2018} - 2) - f(10^{2018} - 1)].$$

We consider each term $f(2n) - f(2n+1)$. Since an even number cannot end in 9, if $2n$ is uphill, then so is $2n+1$, and so $f(2n) - f(2n+1) = 0 - 0 = 0$. If $2n$ is not uphill, then $2n+1$ is one closer than $2n$ to the next uphill integer, so $f(2n) - f(2n+1) = 1$. Therefore, the above sum equals $-N$, where N is the number of non-uphill even integers from 2 to $10^{2018} - 2$, inclusive.

To find N, we instead count the number of uphill even integers from 2 to $10^{2018} - 2$, inclusive; that is, the number of uphill even integers with at most 2018 digits. For $d = 2, 4, 6, 8$, let k be such an integer whose rightmost digit is d. For $i = 1, 2, \ldots, d$, let a_i denote the number of digits of k, besides the rightmost one, which equal i, and note that the tuple $(a_1, a_2, \ldots, a_d)$ uniquely determines k. Since k has at most 2018 digits, we have $a_1 + a_2 + \cdots + a_d \le 2017$, or $a_1 + a_2 + \cdots + a_d + x = 2017$, where x is a nonnegative integer. Thus, by sticks-and-stones, the number of tuples $(a_1, a_2, \ldots, a_d, x)$, and hence the number of values for k, is $\binom{2017+d}{d}$. It follows that the number of even uphill integers is

$$\binom{2019}{2} + \binom{2021}{4} + \binom{2023}{6} + \binom{2025}{8}.$$

So, the number of even non-uphill integers is

$$N = \frac{10^{2018} - 2}{2} - \left(\binom{2019}{2} + \binom{2021}{4} + \binom{2023}{6} + \binom{2025}{8}\right),$$

and so our sum is

$$-N = \frac{2 - 10^{2018}}{2} + \binom{2019}{2} + \binom{2021}{4} + \binom{2023}{6} + \binom{2025}{8}.$$