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**1/4/21.** Archimedes planned to count all of the prime numbers between 2 and 1000 using the Sieve of Eratosthenes as follows:

(a) List the integers from 2 to 1000.
(b) Circle the smallest number in the list and call this $p$.
(c) Cross out all multiples of $p$ in the list except for $p$ itself.
(d) Let $p$ be the smallest number remaining that is neither circled nor crossed out. Circle $p$.
(e) Repeat steps (c) and (d) until each number is either circled or crossed out.

At the end of this process, the circled numbers are prime and the crossed out numbers are composite.

Unfortunately, while crossing off the multiples of 2, Archimedes accidentally crossed out two odd primes in addition to crossing out all the even numbers (besides 2). Otherwise, he executed the algorithm correctly. If the number of circled numbers remaining when Archimedes finished equals the number of primes from 2 to 1000 (including 2), then what is the largest possible prime that Archimedes accidentally crossed out?

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Let $q$ and $r$ be the two primes that Archimedes accidentally crossed out, with $q < r$. Our goal is to maximize $r$ subject to the conditions of the problem.

The composite numbers that will not get crossed out are $q^2$, $qr$, and $r^2$. Since we must have two uncrossed composite numbers between 2 and 1000 (to make up for the 2 primes that accidentally got crossed out), we must have

$$q^2 < qr < 1000 < r^2.$$

Since $q \geq 3$ we must have $r \leq \frac{1000}{3}$. The largest prime number less than $\frac{1000}{3}$ is 331, and indeed, $q = 3$ and $r = 331$ works: if they get crossed out on the first step, then 9 and 993 will not get crossed out and will mistakenly be considered primes. Thus, the answer is $\boxed{331}$.

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**2/4/21.** Let $a$, $b$, $c$, $d$ be four real numbers such that

$$a+b+c+d=8,$$
$$ab+ac+ad+bc+bd+cd=12.$$

Find the greatest possible value of $d$.

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We factor $d$ out of the second equation, and use the fact (from the first equation) that $a+b+c=8-d$:

$$12 = ab+ac+bc+d(a+b+c) = ab+ac+bc+d(8-d). \qquad (\star)$$

However, we know that $a^2+b^2+c^2 \geq ab+ac+bc$: this is an application of what is known as the Rearrangement Inequality, and it also follows by expanding and simplifying $(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$. Thus, we have

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+ac+bc) \geq 3(ab+ac+bc), \qquad (\star\star)$$

giving

$$ab+ac+bc \leq \frac{1}{3}(a+b+c)^2 = \frac{1}{3}(8-d)^2.$$

Therefore, $(\star)$ becomes

$$12 \leq \frac{1}{3}(8-d)^2 + d(8-d),$$

which simplifies to $d^2-4d-14 \leq 0$. The largest value of $d$ satisfying this inequality is the larger root of $d^2-4d-14=0$, which is $d=2+3\sqrt{2}$. We verify that equality holds in $(\star\star)$ when $a=b=c$, giving

$$a=b=c=\frac{1}{3}(8-d)=\frac{1}{3}(6-3\sqrt{2})=2-\sqrt{2},$$

and it is easily checked that

$$ab+ac+ad+bc+bd+cd = 3(2-\sqrt{2})^2+3(2-\sqrt{2})(2+3\sqrt{2}) = 3((6-4\sqrt{2})+(-2+4\sqrt{2})) = 12,$$

as necessary. Therefore, the answer is $\boxed{2+3\sqrt{2}}$.

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**3/4/21.** I give you a deck of $n$ cards numbered 1 through $n$. On each turn, you take the top card of the deck and place it anywhere you choose in the deck. You must arrange the cards in numerical order, with card 1 on top and card $n$ on the bottom. If I place the deck in a random order before giving it to you, and you know the initial order of the cards, what is the expected value of the minimum number of turns you need to arrange the deck in order?

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$$\boxed{\begin{array}{c}3\\6\\5\\1\\2\\4\end{array}}$$

In any arrangement, there will be $k$ cards at the bottom (where $1 \le k \le n$) that are in ascending numerical order, such that the $(k+1)$ cards at the bottom (if $k < n$) are not in ascending numerical order. (If $k = n$ then the entire deck is already in correct order, and 0 turns are required.) For example, if the deck is arranged as at right (for $n = 6$), then $k = 3$ since the 3 bottom cards are in ascending order—$1, 2, 4$—but the 4 bottom cards—$5, 1, 2, 4$—are not. Since the $(k + 1)^{\text{st}}$ card must be moved to its correct position, all of the cards above the bottom $k$ cards must be moved at least once. We claim that this is sufficient: on the $i^{\text{th}}$ turn, move the top card to its correct position among the bottom $k - 1 + i$ cards (which by assumption are already in the proper order). Thus $n - k$ turns are required. For example, below we show the sequence of moves for the deck at right: note that after $i$ turns the bottom $3+i$ cards are in order, but the card immediately above this group is not (the cards at the bottom in correct order are boxed):

$$\begin{array}{c}3\\6\\5\\\boxed{1}\\\boxed{2}\\\boxed{4}\end{array}\;\rightarrow\;\begin{array}{c}6\\5\\\boxed{1}\\\boxed{2}\\\boxed{3}\\\boxed{4}\end{array}\;\rightarrow\;\begin{array}{c}5\\\boxed{1}\\\boxed{2}\\\boxed{3}\\\boxed{4}\\\boxed{6}\end{array}\;\rightarrow\;\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\\\boxed{4}\\\boxed{5}\\\boxed{6}\end{array}$$

Let $E_n$ be the expected number of turns required for a deck of size $n$. We complete the solution via two different methods:

*Method 1*: Unless the deck is already in correct order, we must move the top card. What remains, disregarding the card that we just moved (which is now in its correct position in the group at the bottom of the deck) is a deck of size $n-1$, which will take an expected $E_{n-1}$ turns to arrange correctly. So we require an expected $1 + E_{n-1}$ number of turns. However, there is a $\frac{1}{n!}$ probability that the deck will start in its correct order, in which case we will not even require the first turn. Thus, we have

$$E_n = 1 + E_{n-1} - \frac{1}{n!}.$$

Starting with $E_1 = 0$, we have

$$E_n = \boxed{(n-1) - \sum_{k=2}^{n} \frac{1}{k!}} = \boxed{n - \sum_{k=1}^{n} \frac{1}{k!}}.$$

*Method 2*: Suppose the $k$ cards at the bottom are in ascending numerical order, but the $(k+1)$ cards at the bottom are not, so that $n-k$ turns are required. To count the number of such initial arrangements, we note that for each of the $\binom{n}{k+1}$ choices of $k+1$ cards, there are $k$ ways to order the cards such that the final $k$ are in ascending order, but the final $k+1$ are not (all such arrangements can be formed by starting with the $k+1$ cards in order, then moving one of the final $k$ cards to the top of this group of $k+1$ cards). There are then $(n-(k+1))!$ ways to order the remaining cards atop these $k+1$ cards. So, there are $k(n-(k+1))!\binom{n}{k+1}$ such arrangements, and we require $n-k$ turns for each.

Summing over all possible $k$, we have

$$E_n = \frac{\sum_{k=1}^{n-1}(n-k)k(n-(k+1))!\binom{n}{k+1}}{n!} = \frac{\sum_{k=1}^{n-1}(n-k)k\frac{n!}{(k+1)!}}{n!} = \sum_{k=1}^{n-1}\frac{nk-k^2}{(k+1)!}.$$

We can write this in a more appealing form with some manipulation that leads to telescoping series:

$$\begin{aligned}
E_n &= \sum_{k=1}^{n-1}\frac{nk-k^2}{(k+1)!}\\
&= \sum_{k=1}^{n-1}\frac{n(k+1)-n-k(k+1)+k}{(k+1)!}\\
&= \sum_{k=1}^{n-1}\frac{n(k+1)-n-k(k+1)+(k+1)-1}{(k+1)!}\\
&= \sum_{k=1}^{n-1}\left(\frac{n}{k!}-\frac{n}{(k+1)!}\right)+\sum_{k=1}^{n-1}\left(-\frac{1}{(k-1)!}+\frac{1}{k!}\right)-\sum_{k=1}^{n-1}\frac{1}{(k+1)!}.
\end{aligned}$$

The first two summations telescope, and we are left with

$$\begin{aligned}
E_n &= \frac{n}{1!}-\frac{n}{n!}-\frac{1}{0!}+\frac{1}{(n-1)!}-\sum_{k=1}^{n-1}\frac{1}{(k+1)!}\\
&= n-\frac{1}{(n-1)!}-1+\frac{1}{(n-1)!}-\sum_{k=1}^{n-1}\frac{1}{(k+1)!}\\
&= \boxed{n-\sum_{k=1}^{n}\frac{1}{k!}}.
\end{aligned}$$

Note: there is no closed form for the above summation, but, using calculus, we can show that $E_n$ is very close to $n+1-e$ as $n$ gets large, where $e \approx 2.71828\ldots$ is the base of the natural logarithm.

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**4/4/21.** Let $S$ be a set of 10 distinct positive real numbers. Show that there exist $x, y \in S$ such that

$$0 < x - y < \frac{(1+x)(1+y)}{9}.$$

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Let the set be $\{x_1, x_2, \ldots, x_{10}\}$, where $0 < x_1 < x_2 < \cdots < x_{10}$. Let

$$a_i = \frac{9}{x_i + 1},$$

so $0 < a_{10} < a_9 < \cdots < a_1 < 9$ and $x_i = \dfrac{9 - a_i}{a_i}$.

Let $i < j$. Then $x_i < x_j$, and

$$\begin{aligned} x_j - x_i \le \frac{(1+x_i)(1+x_j)}{9} \quad &\Leftrightarrow \quad \frac{9-a_j}{a_j} - \frac{9-a_i}{a_i} \le \frac{\frac{9}{a_i} \cdot \frac{9}{a_j}}{9} \\ &\Leftrightarrow \quad \frac{9a_i - 9a_j}{a_i a_j} \le \frac{9}{a_i a_j} \\ &\Leftrightarrow \quad a_i - a_j \le 1. \end{aligned}$$

Suppose that no such $x$ and $y$ exist that satisfy the given condition. Then $a_1 - a_2 > 1$, $a_2 - a_3 > 1$, $\ldots$, $a_9 - a_{10} > 1$. Adding these inequalities, we get $a_1 - a_{10} > 9$. But $a_1 < 9$ and $a_{10} > 0$, so $a_1 - a_{10} < 9$, a contradiction. Therefore, such $x$ and $y$ must exist.

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**5/4/21.** Tina and Paul are playing a game on a square $\mathcal{S}$. First, Tina selects a point $T$ inside $\mathcal{S}$. Next, Paul selects a point $P$ inside $\mathcal{S}$. Paul then colors blue all the points inside $\mathcal{S}$ that are closer to $P$ than $T$. Tina wins if the blue region thus produced is the interior of a triangle. Assuming that Paul is lazy and simply selects his point at random (and that Tina knows this), find, with proof, a point Tina can select to maximize her probability of winning, and compute this probability.

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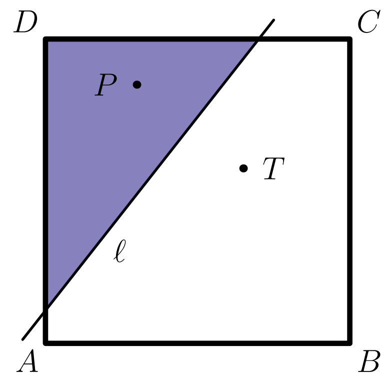
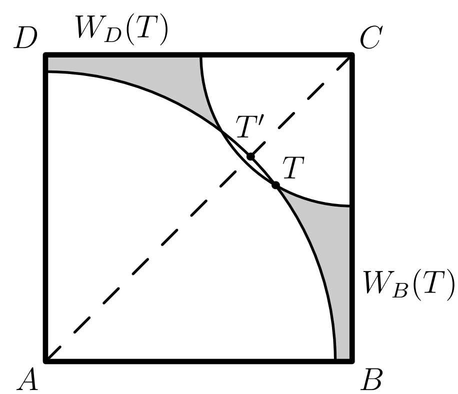



Let the square be $ABCD$, and for simplicity assume it has side length 1. Let Tina's point be $T$ and Paul's be $P$, and let the perpendicular bisector of $\overline{PT}$ be $\ell$. All points on the same side of $\ell$ as $T$ are closer to $T$ than to $P$, and conversely all points on the same side of $\ell$ as $P$ are closer to $P$ than to $T$. Thus, the blue region is the intersection of the interior of the square with the half-plane containing $P$ determined by $\ell$. This is a triangle if and only if exactly 1 of $\{A, B, C, D\}$ is in this half-plane, meaning that Tina wins if and only if $T$ is closer than $P$ to three of the four vertices of the square. This occurs if and only if $T$ is closer than $P$ to either pair of opposite vertices (either $A$ and $C$, or $B$ and $D$).

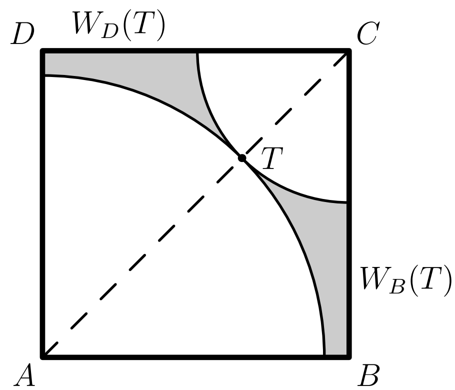



Consider the circle centered at $A$ with radius $AT$ and the circle centered at $C$ with radius $CT$; we call these $\Gamma_A(T)$ and $\Gamma_C(T)$. The point $T$ is closer than $P$ to both $A$ and $C$ if and only if $P$ is outside both circles, since such a location of $P$ gives $PA > TA$ and $PC > TC$. We call the two regions inside the square, but outside both circles, "winning regions," because Tina wins if Paul chooses a point in either of these regions. There is a winning region $W_B(T)$ of points $P$ for which $P$ is closer to $B$ than $T$ but where $T$ is closer to each of $A$, $C$, and $D$ than $P$, as shown in the diagram. Similarly, there is a winning region $W_D(T)$. Because the diagram is symmetric about $\overline{AC}$, we have $W_B(T) \cong W_D(T)$. There is a similar pair of winning regions $W_A(T)$ and $W_C(T)$ in which $T$ is closer than $P$ to both $B$ and $D$.

We claim that Tina maximizes the total area of the winning regions by choosing $T$ to be the center of the square. We show this separately for the pair $A, C$ and the pair $B, D$, in two steps.

For any choice of $T$ not on $\overline{AC}$, let $T'$ be the intersection of $\overline{AC}$ and $\Gamma_A(T)$. We claim that $W_B(T') \supset W_B(T)$ and $W_D(T') \supset W_D(T)$. This is clear because $\Gamma_C(T')$ lies inside

$\Gamma_C(T)$, and hence the winning regions of $T'$ are larger than those of $T$, since they contain an additional region that is inside $\Gamma_C(T)$ but outside $\Gamma_C(T')$. Thus, we conclude that for any point $T$ not on $\overline{AC}$, there is a point $T' \in \overline{AC}$ such that the total area of $W_B(T')$ and $W_D(T')$ is greater than the total area of $W_B(T)$ and $W_D(T)$. Therefore, the point that maximizes the winning regions adjacent to $B$ and $D$ must lie on the diagonal $\overline{AC}$.




Next, we show that of all points on $\overline{AC}$, Tina maximizes the area of the winning regions $W_B(T)$ and $W_D(T)$ by choosing the center of the square. Note that if $T \in \overline{AC}$, then the area of the winning regions is the entire area of the square minus the areas of the portions of $\Gamma_A(T)$ and $\Gamma_C(T)$ lying inside the square (the white areas in the diagram). So our goal is to minimize the white areas. If $AT$ and $CT$ are both less than or equal to 1, this white region is two quarter-circles whose radii sum to the diagonal of the square, which is $\sqrt{2}$. Thus, if the radius of $\Gamma_A(T)$ is $r$, then the radius of $\Gamma_C(T)$ is $\sqrt{2} - r$, and the combined area of the white regions is $\frac{\pi}{4}(r^2 + (\sqrt{2} - r)^2) = \frac{\pi}{4}(2r^2 - 2\sqrt{2}r + 2)$. This quadratic is minimized at $r = \frac{\sqrt{2}}{2}$, which is when each radius is half the diagonal, or $\sqrt{2}/2$, giving a white area of

$$2 \cdot \frac{1}{4} \cdot \pi \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{4}.$$

If $AT > 1$ or $CT > 1$, then the corresponding white region is not a quarter circle; however, in this case the white region is larger than a quarter-circle of radius 1, which has area $\frac{\pi}{4}$, so the white region is still larger than the minimum area.

Since the center point of the square maximizes the sum of the areas of the winning regions $W_B(T)$ and $W_D(T)$, and by symmetry simultaneously maximizes the sum of the areas of the winning regions $W_A(T)$ and $W_C(T)$, and these regions are all disjoint, we conclude that the center point is the optimal point for Tina to select. As computed above, the regions $W_B(T)$ and $W_D(T)$ have a total area of $1 - \frac{\pi}{4}$, as do the regions $W_A(T)$ and $W_C(T)$, so the total winning area for Tina is $2(1 - \frac{\pi}{4}) = 2 - \frac{\pi}{2}$. Thus, since the total area of the square is 1, the probability of Tina winning is just the total area of the winning regions, or $\boxed{2 - \frac{\pi}{2}}$.

*Credits: Problem statements and solutions were written by USAMTS staff.*
*Problem 2/4/21 was submitted by Phan Van Thuan, Vietnam National University.*
*Problem 3/4/21 was originally suggested by Matt Superdock.*