USA Mathematical Talent Search

Round 1 Solutions

Year 21 — Academic Year 2009–2010

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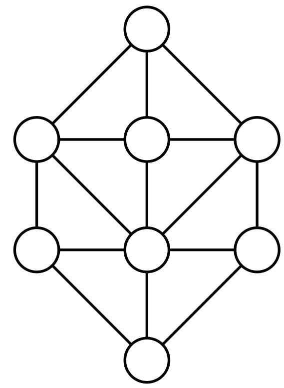
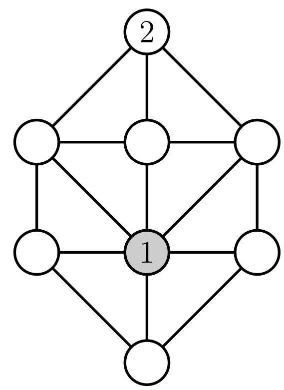
1/1/21. Fill in the circles in the picture at right with the digits 1-8, one digit in each circle with no digit repeated, so that no two circles that are connected by a line segment contain consecutive digits. In how many ways can this be done?



We observe first that the lower center circle (shaded in the diagram below to the right) is connected to all but one of the other circles. If we place any digit other than 1 or 8 in this circle we will not be able to complete the grid, since the digits 2–7 each have two other digits that are not allowed to be adjacent to it. Furthermore, if the lower center circle contains 1, then the top circle must contain 2, because 2 cannot be placed adjacent to 1.

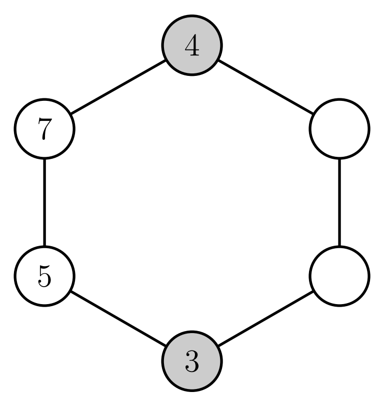


We notice that if we are given a solution to this problem, then replacing every number x in the graph by $9-x$ gives us a different solution. Furthermore this operation gives a 1-1 correspondence between the solutions with 1 in the lower center circle and the solutions with 8 in the lower center circle. Thus, we count the number of solutions where 1 is in this circle, and the total will be twice this count.

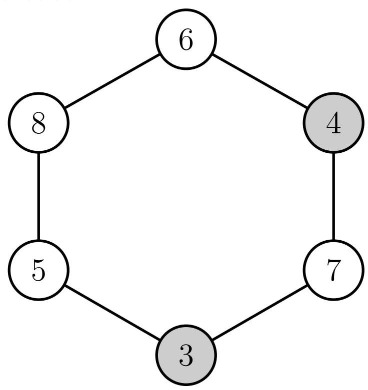
The remaining six empty circles form a hexagon. We must place the digit 3 in one of the bottom 3 circles (since it cannot be adjacent to 2). Once we have chosen where to place 3, we are left filling the hexagon with the digits 4 through 8. At this point the locations of 1 and 2 are irrelevant, so we can count each of these three possibilities for the location of 3 by counting three times the number of possibilites when 3 is placed at the bottom of a hexagon and the remaining digits are placed elsewhere. We now see that the answer to the original problem is $2 \cdot 3 = 6$ times the number of ways to fill a six-circle loop with the digits 3 to 8 such that 3 is on the bottom and no two consecutive digits are adjacent.

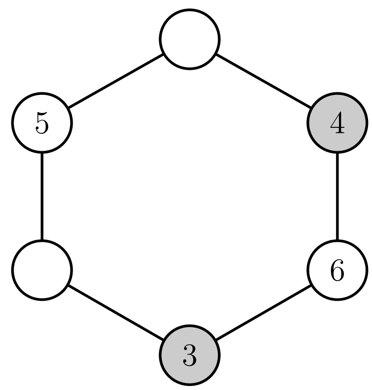
Next we place the digit 4. There are two cases:

Case 1: 4 is opposite from 3. From here we have two choices for the location of 5 (either of the circles adjacent to 3), and that determines where 7 is placed uniquely, as in the figure to the right. At this point we can place 6 and 8 in either of other two nodes, giving a total of $2 \cdot 2 = 4$ possibilities for this case.



Case 2: 4 is two positions away from 3. There are two choices for the location of 4, and these are symmetric, so we assume 4 is on the right-hand side of the diagram (and we remember that we will get a factor of two from this symmetry). Now we have two non-symmetric choices for placing 5. If we place 5 adjacent to 3 as in the left image below, there is only one way to complete the diagram, as shown. If we place 5 a distance two from 3 (as in the picture to the right below), then the location of 6 is now determined (as shown) and 7 and 8 may be placed freely. This gives two more options. So there are a total of $2 \cdot (1+2) = 6$ possibilities in this case.





Summing the two cases, this gives a total of $4 + 6 = 10$ possibilities once 3 is placed. Recalling the factor of 6 from earlier, we conclude that the total number of possibilities is $6 \cdot 10 = \boxed{60}$.

2/1/21. The ordered pair of four-digit numbers $(2025, 3136)$ has the property that each number in the pair is a perfect square and each digit of the second number is 1 more than the corresponding digit of the first number. Find, with proof, all ordered pairs of five-digit numbers and ordered pairs of six-digit numbers with the same property: each number in the pair is a perfect square and each digit of the second number is 1 more than the corresponding digit of the first number.

We observe first that if the digits of n^2 are one less than the digits of m^2, then $m^2 - n^2 = 11111$ or $m^2 - n^2 = 111111$ (depending on the lengths of m^2 and n^2). Thus we start by looking for solutions to these two equations and then test all of our solutions.

If (n^2, m^2) is an ordered pair of 5-digit numbers satisfying the desired property, then we must have

$$11111 = m^2 - n^2 = (m - n)(m + n)$$

The number 11111 has only two factorizations into a product of two factors: $11111 = 41 \cdot 271$ and $11111 = 1 \cdot 11111$. Checking the two factorizations we have

$$(n, m) = (115, 156)$$
$$(n, m) = (5550, 5551)$$

The second pair will fail since $n > 1000$ implies that n^2 must have at least 7 digits. We check the first:

$$(n^2, m^2) = (13225, 24336).$$

This pair works. So the only 5-digit pair is $(13225, 24336)$.

Next we look for ordered pairs (n^2, m^2) of six-digit numbers with the desired property. We must have

$$111111 = m^2 - n^2 = (m - n)(m + n).$$

Again we factor: $111111 = 3 \times 7 \times 11 \times 13 \times 37$. Thus, 111111 has $2^5 = 32$ divisors, so there are 16 ways to factor 111111 into the product of a pair of positive integers. But we must have $100000 \le n^2 \le m^2 \le 999999$, so this severely restricts the possibilities: taking the square root, we must have

$$317 \le n < m \le 999.$$

This means that the larger factor $m+n$ must be at least 635 but at most 1997. This restricts the choice of $(m - n, m + n)$ to the following pairs:

$$\{(143, 777), (111, 1001), (91, 1221), (77, 1443)\}.$$

These correspond to (n, m) equaling one of

$$\{(317, 460), (445, 556), (565, 656), (683, 760)\}.$$

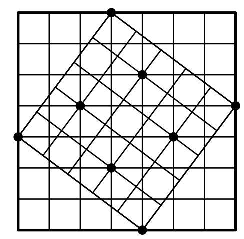
We check these by computing (n^2, m^2):

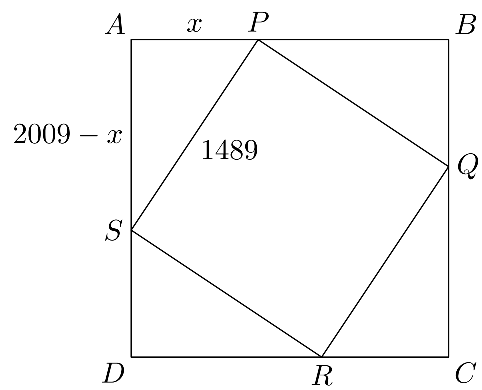
$$\{(100489, 211600), (198025, 309136), (319225, 430336), (466489, 577600)\}.$$

None of these pairs satisfies the desired property.

Our answer is:
There is one pair of 5-digit numbers which satisfies the property, $\boxed{(13225, 24336)}$.
There are $\boxed{\text{no pairs}}$ of 6-digit numbers which satisfy the property.

3/1/21. A square of side length 5 is inscribed in a square of side length 7. If we construct a grid of 1×1 squares for both squares, as shown to the right, then we find that the two grids have 8 lattice points in common. If we do the same construction by inscribing a square of side length 1489 in a square of side length 2009, and construct a grid of 1×1 squares in each large square, then how many lattice points will the two grids of 1×1 squares have in common?





Let the two squares be $ABCD$ and $PQRS$, and let $x = AP = BQ = CR = DS$. Then $PB = QC = RD = SA = 2009 - x$. Then by the Pythagorean Theorem on triangle APS,

$$x^2 + (2009 - x)^2 = 1489^2,$$

which has solutions $x = 689$ and $x = 1320$. The lengths 689 and 1320 produce equivalent diagrams, so assume $x = 689$.

For brevity, let $a = 689$, $b = 1320$, and $c = 1489$; note that c is prime. Consider the diagram on the coordinate plane with D as the origin, so that $C = (2009, 0)$ and $A = (0, 2009)$. Then, by thinking of starting at point S and moving in units in the directions of $\overrightarrow{SR}$ and $\overrightarrow{SP}$, we see that points on the lattice of $PQRS$ are points of the form

$$(0, a) + u\left(\frac{b}{c}, -\frac{a}{c}\right) + v\left(\frac{a}{c}, \frac{b}{c}\right)$$

where $0 \le u \le c$ and $0 \le v \le c$. This is a point on both lattices if and only if the coordinates are integers; that is, if

$$\frac{ub + va}{c} \text{ and } \frac{-ua + vb}{c}$$

are both integers. So our goal is to count all (u, v) with $0 \le u \le c$ and $0 \le v \le c$ such that $ub + va$ and $-ua + vb$ are integer multiples of c; that is,

$$ub + va \equiv 0 \pmod{c}, \qquad (1)$$
$$-ua + vb \equiv 0 \pmod{c}. \qquad (2)$$

Clearly if $(u, v) \in \{(0, 0), (0, c), (c, 0), (c, c)\}$ then this condition is satisfied—these correspond to the four corners of the smaller square. We claim that if $0 < u < c$, then there exists a unique v with $0 < v < c$ satisfying (1) and (2). In particular, since c is prime, equation (1) can be rewritten as

$$v \equiv -\frac{ub}{a} \pmod{c},$$

where "division by a" is well-defined modulo c. (Indeed, with $a = 689, c = 1489$ we can check that $\frac{1}{a} \equiv 389 \pmod{c}$, since $(389)(689) = 268021 = (180)(1489) + 1$.) So there is a unique v with $0 < v < c$ satisfying (1). To verify that this also satisfies (2), note that since $a^2 + b^2 = c^2$, we have $b^2 \equiv -a^2 \pmod{c}$, and thus

$$-ua + vb \equiv -ua + \left(-\frac{ub}{a}\right) b \equiv -\left(a + \frac{b^2}{a}\right) u \equiv -\left(a + \frac{-a^2}{a}\right) u \equiv 0 \pmod{c}.$$

Thus, each $0 < u < c$ gives exactly one ordered pair (u, v) that corresponds to a lattice point of both grids. This gives $c - 1$ such points, so together with the four corners of $PQRS$, we get a total of $c + 3$ points. Therefore, since $c = 1489$, the answer is $\boxed{1492}$.

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4/1/21. Let $ABCDEF$ be a convex hexagon, such that $FA = AB$, $BC = CD$, $DE = EF$, and $\angle FAB = 2\angle EAC$. Suppose that the area of ABC is 25, the area of CDE is 10, the area of EFA is 25, and the area of ACE is x. Find, with proof, all possible values of x.

Since $\angle FAB = 2\angle EAC$, and the hexagon is convex, we have

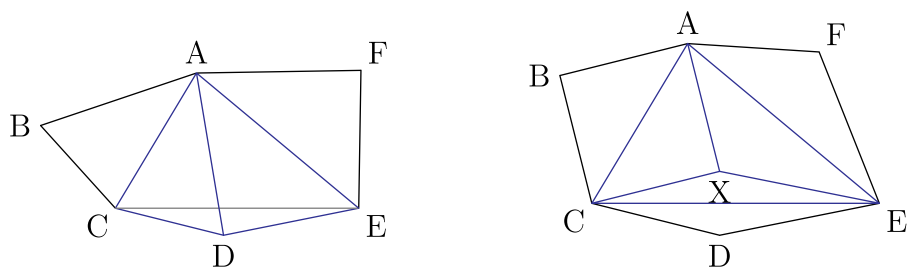
$$\angle EAC = \angle FAB - \angle EAC = \angle FAE + \angle BAC.$$

Let X be the reflection of B across AC. Then $CX = CB = CD$. Also, $AX = AB = AF$, and

$$\angle XAE = \angle EAC - \angle XAC = (\angle FAE + \angle BAC) - \angle BAC = \angle FAE.$$

Since $AX = AF$ and $\angle XAE = \angle FAE$, we conclude that X is also the reflection of F across AE. Hence $EX = EF = ED$.

Since $CX = CD$ and $EX = ED$, we have two possibilities: either $X = D$, or X is the reflection of D across CE. Both cases are shown below.



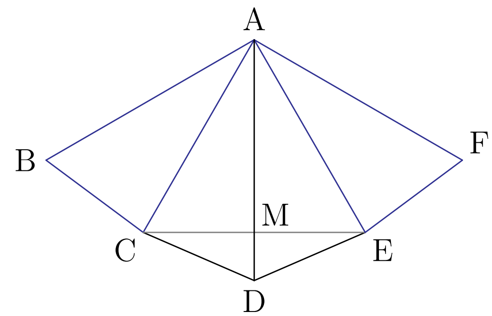
Let $[PQR]$ denote the area of triangle PQR. In the first case,

$$[ACE] = [ADC] + [ADE] - [CDE] = [ABC] + [AFE] - [CDE] = 25 + 25 - 10 = 40.$$

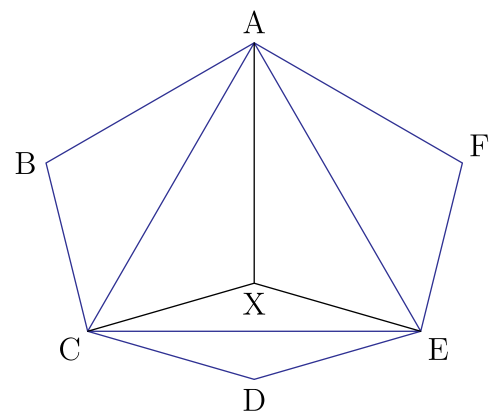
In the second case,

$$[ACE] = [AXC] + [AXE] + [CXE] = [ABC] + [AFE] + [CDE] = 25 + 25 + 10 = 60.$$

Hence, the area of triangle ACE must be either 40 or 60. However we have not yet shown that we can achieve these values. It remains to construct convex hexagons with the initial conditions such that triangle ACE has these areas.



For the first case, let ACE be an equilateral triangle with area 40, and let M be the midpoint of CE. Let $CM = a$, so $AM = a\sqrt{3}$. Let D be the point on the perpendicular bisector of CE, on the opposite side of CE from A, such that $DM = \frac{a\sqrt{3}}{4}$. Since triangle CDE shares a base with triangle ACE but its altitude to that base is 1/4 the altitude of ACE, its area is 1/4 that of ACE; thus, $[CDE] = \frac{1}{4}(40) = 10$. Reflect D across AC to get B, and reflect D across AE to get F. We have $[ABC] = [ADC] = [AMC] + [DMC] = \frac{1}{2}(40) + \frac{1}{2}(10) = 25$, and similarly $[AFE] = 25$. Also, $\tan\angle MCD = \frac{a(\sqrt{3}/4)}{a} = \frac{\sqrt{3}}{4} < \frac{\sqrt{3}}{3} = \tan 30^\circ$, so $\angle MCD < 30^\circ$. Thus, $\angle BCD = 2\angle ACD = 2(\angle ACM + \angle MCD) < 2(60^\circ + 30^\circ) = 180^\circ$. Similarly, $\angle FED < 180^\circ$, and $\angle FAB = 2\angle EAC = 2(60^\circ) < 180^\circ$. The remaining angles are obviously less than 180°, so hexagon $ABCDEF$ is convex. We have thus constructed the first case.



For the second case, let ACE be an equilateral triangle with area 60, and let M be the midpoint of CE. Let X be the point on segment AM such that $XM = \frac{1}{6}AM$. Then $[CXE] = \frac{1}{6}[ACE] = \frac{1}{6}(60) = 10$. Thus, $[AXC] = \frac{1}{2}([ACE]-[CXE]) = \frac{1}{2}(60-10) = 25$, and similarly $[AXE] = 25$. Let B, D, and F be the reflections of X across AC, CE, and EA, respectively. Then $[ABC] = 25$, $[CDE] = 10$, and $[EFA] = 25$. Also, $\angle FAB = 2\angle EAC = 2(60^\circ) < 180^\circ$, and similarly $\angle BCD < 180^\circ$ and $\angle DEF < 180^\circ$. The remaining angles are obviously less than 180°, so hexagon $ABCDEF$ is convex. Thus, we have constructed the second case.

Therefore, we conclude that $\boxed{40 \text{ and } 60}$ are the two possible areas of triangle ACE.

5/1/21. The cubic equation $x^3+2x-1=0$ has exactly one real root r. Note that $0.4 < r < 0.5$.

(a) Find, with proof, an increasing sequence of positive integers $a_1 < a_2 < a_3 < \cdots$ such that
$$\frac{1}{2} = r^{a_1} + r^{a_2} + r^{a_3} + \cdots .$$

(b) Prove that the sequence that you found in part (a) is the unique increasing sequence with the above property.

(a) Since r is a root of $x^3 + 2x - 1$, we know that $1 - r^3 = 2r$, so
$$\frac{1}{2} = \frac{r}{1-r^3} = r(1 + r^3 + r^6 + r^9 + \cdots) = r + r^4 + r^7 + r^{10} + \cdots$$
Furthermore, since $|r| < 1$, this sequence converges. Therefore $\boxed{a_n = 3n - 2}$ solves the equation above.

(b) Suppose that $a_1 < a_2 < a_3 < \cdots$ and $b_1 < b_2 < b_3 < \cdots$ are distinct sequences of positive integers such that
$$r^{a_1} + r^{a_2} + r^{a_3} + \cdots = r^{b_1} + r^{b_2} + r^{b_3} + \cdots = \frac{1}{2}.$$
Eliminating duplicate terms, we have
$$r^{s_1} + r^{s_2} + \cdots = r^{t_1} + r^{t_2} + \cdots = a > 0,$$
where $s_1 < s_2 < \ldots$ and $t_1 < t_2 < \ldots$ are positive integers, and all the s's and t's are distinct. (Note that either series above might in fact be finite, but this does not affect the validity of the argument to follow.) Assume with loss of generality that $s_1 < t_1$. Then
$$r^{s_1} \le r^{s_1} + r^{s_2} + \cdots = r^{t_1} + r^{t_2} + \ldots,$$
so that, after dividing by r^{s_1}, we have
$$1 \le r^{u_1} + r^{u_2} + \ldots,$$
where $u_i = t_i - s_1 > 0$. Thus, since $0 < r < \frac{1}{2}$, we have $\frac{1}{1-r} < 2$, and hence
$$1 \le r^{u_1} + r^{u_2} + \cdots \le r + r^2 + r^3 + \cdots = \frac{1}{1-r} - 1 < 1,$$
a contradiction.

Credits: Problem 2/1/21 is based on a problem appearing in the February, 1997, issue of Crux Mathematicorum with Mathematical Mayhem.
Problem 4/1/21 was proposed by Gaku Liu.
All other problems and solutions by USAMTS staff.