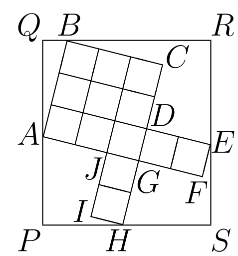
4/3/16. Region $ABCDEFGHIJ$ consists of 13 equal squares and is inscribed in rectangle $PQRS$ with A on $\overline{PQ}$, B on $\overline{QR}$, E on $\overline{RS}$, and H on $\overline{SP}$, as shown in the figure on the right. Given that $PQ = 28$ and $QR = 26$, determine, with proof, the area of region $ABCDEFGHIJ$.

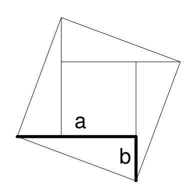


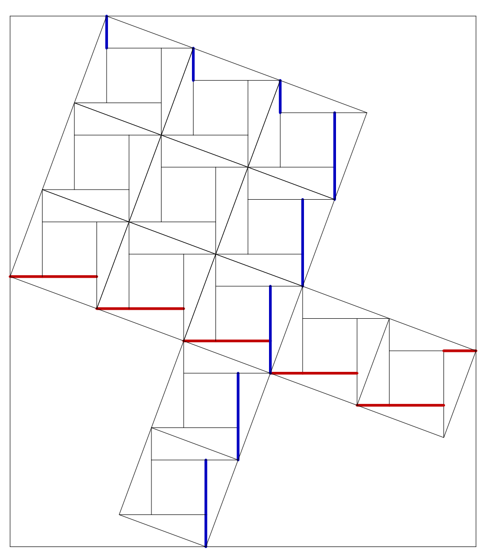
Credit This problem was inspired by Problem 2 of the First Round of the 2001 Japanese Mathematical Olympiad.

Comments There were many different successful approaches to solving this problem. The most aesthetically pleasing comes from Zachary Abel (11/TX). Others took a similar approach, using projection without Zachary's clever dissection. Noah Cohen gives an example of this approach. Nicholas Zehender shows that a subset of the figure formed by the little squares could itself be inscribed in a square and used this fact to solve the problem. Finally, Michael John Griffin gives us another dissection to solve the problem.

Solution 1 by: Zachary Abel (11/TX)

Inside each square in the diagram, draw two horizontal segments and two vertical segments as shown to the right. Let the two indicated lengths be a and b. The whole diagram looks like this:





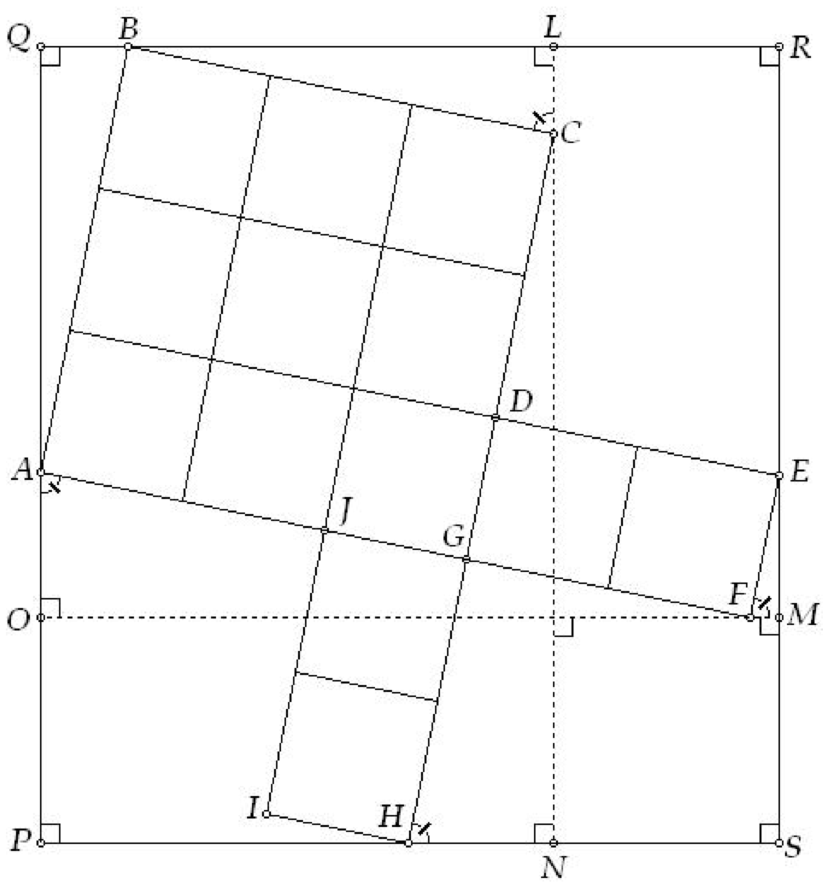
The total horizontal length of the red segments is $5a + b$, which is equal to the width of the rectangle, i.e. $5a + b = 26$. Likewise, the total vertical length of the blue segments is

$5a + 3b$, which is equal to the height of the rectangle, 28. So we have the system

$$\begin{cases} 5a + b &= 26 \\ 5a + 3b &= 28 \end{cases},$$

whose solution is $a = 5$ and $b = 1$. So the side length of each square is $\sqrt{a^2 + b^2} = \sqrt{26}$, the area of each square is 26, and the total area of $ABCDEFGHIJ$ is $26 \times 13 = 338$.

Solution 2 by: Noah Cohen (11/ME)



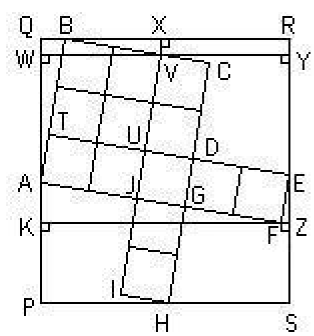
Via angle chasing, it can be seen that $\angle BCL \equiv \angle GHN \equiv \angle OAJ \equiv \angle EFM$, call this angle ϑ, and call the length of one square x

We can now represent the height and and width of the rectangle with the following equations

$$\begin{aligned} 5x\sin\vartheta + x\cos\vartheta &= 26 \\ 5x\sin\vartheta + 3x\cos\vartheta &= 28 \\ x\cos\vartheta &= 1 \\ x\sin\vartheta &= 5 \\ (x\sin\vartheta)^2 + (x\cos\vartheta)^2 &= x^2 \\ x &= \sqrt{26} \end{aligned}$$

The area of the polygon $ABCDEFGHIJ$ can be represented by $13x^2$, or $(13)(\sqrt{26})^2$, which is equal to 338.

Solution 3 by: Nicholas Zehender (11/VA)



Draw a line parallel to QR through point V. $CDEFGHIJATUV$ is the same if you rotate it 90 degrees, so $YS = WY = QR = 26$. $RY = RS - YS = 28 - 26 = 2$, so $XV = 2$. $AFK \sim VBX$ because $AF \parallel VB$, $FK \parallel BX$, and $KA \parallel XV$. $\angle EFZ = 180 - 90 - \angle AFK = 90 - \angle AFK = \angle FAK$, and $\angle EZF = \angle FKA = 90$, so FEZ is also similar to AFK and VBX.

$ZF/XV = EF/BV$
$ZF/2 = 1/2$
$ZF = 1$

$FK = ZK - ZF = 26 - 1 = 25$

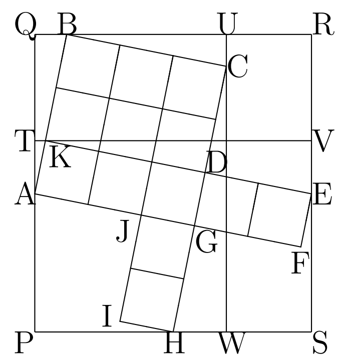
$KA/XV = AF/BV$
$KA/2 = 5/2$
$KA = 5$

$AF = \sqrt{FK^2 + KA^2}$
$AF = \sqrt{625 + 25}$
$AF = \sqrt{650}$

The side of one of the squares is $AF/5 = \sqrt{26}$, so the area of $ABCDEFGHIJ$ is $13(\sqrt{26})^2 = 338$.

Solution 4 by: Michael John Griffin (12/UT)

Let Point K be the point one third of the way between points A and B so that it is also in line with point E. Also, let points T, U, V, and W be points along lines $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, and $\overline{SP}$ respectively, such that line $\overline{TV}$ is perpendicular to line $\overline{PQ}$ and passes through point K, and line $\overline{UW}$ is perpendicular to line $\overline{QR}$ and passes through point C, as shown in the figure at right.



Triangles ΔABQ, ΔBCU, ΔAKT, ΔKEV, and ΔCHW are all similar, with $\Delta ABQ \cong \Delta BCU$ and $\Delta KEV \cong \Delta CHW$. All of these triangles have one right angle and the other corresponding angles equal. If two given angles (such as $\angle ABQ$ and $\angle CBU$) and a right angle are all collinear, the two angles are complimentary. In the case of ΔAKT and ΔABQ, Euclid's Coresponding Angles postulate works well.

$\overline{KV} = \overline{CW}$, $\overline{UC} = \overline{QB}$, and $\overline{TK} = 1/3\overline{QB}$ (given the ratio of the hypotenuses similar triangles). Notice that $\overline{UC} + \overline{CW} = 28$ and $\overline{TK} + \overline{KV} = 26$.

$$\begin{aligned}
\overline{UC} + \overline{CW} &= \overline{TK} + \overline{KV} + 2 \\
\overline{QB} + \overline{CW} &= \frac{\overline{QB}}{3} + \overline{CW} + 2 \\
\frac{2 * \overline{QB}}{3} &= 2 \\
\overline{QB} &= 3
\end{aligned}$$

Since $\overline{UC} = \overline{QB}$ and $\overline{UC} + \overline{CW} = 28$, $\overline{CW} = 25$. $\overline{CW} = \frac{5*\overline{AQ}}{3}$, so $\overline{AQ} = 15$. $\overline{AQ}^2 + \overline{QB}^2 = \overline{AB}^2$, so $\overline{AB}^2 = 15^2 + 3^2 = 225 + 9 = 234$. $\overline{AB}^2$ just happens to be the area of 9 of the 13 squares, so the total area of all the squares is $13/9 * 234 = 338\ units^2$