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Solutions to Problem 4/1/16
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4/1/16. The interior angles of a convex polygon form an arithmetic progression with a common difference of 4°. Determine the number of sides of the polygon if its largest interior angle is 172°.

Credit This problem was modeled after Problem 106 on page 75 of Volume 3, Number 1 (Spring 1981) of AGATE, which was edited in Alberta, Canada, by Professor Andy Liu, a long-time friend of the USAMTS program. Dr. Liu referenced the *Second Book of Mathematical Bafflers*, edited by A. F. Dunn and published by Dover Publications in 1983.

Comments The most straightforward solution is as in Solution 1. A slightly more elegant solution, using exterior angles instead of interior angles, is shown in Solution 2. Note that both method leads to the same quadratic equation. It is also possible to use a "guess and check" method on this problem.

Solution 1 by: Yakov Berchenko-Kogan (10/NC)

Let n be the number of sides of the polygon. The first step to solving this problem is to determine the sum of the interior angles determined by the arithmetic progression in terms of n. It is fairly clear that the progression is $172, 168, 164, ..., 172 - 4(n-1)$. Thus we can find its sum to be:

$$\begin{aligned} & 172n - 4\frac{n(n-1)}{2} \\ = \;& 172n - 2n(n-1) \\ = \;& 174n - 2n^2. \end{aligned}$$

We also know the formula for the sum of the interior angles of an n-sided polygon: $180(n-2) = 180n - 360$. Thus we can equate these two and solve for n:

$$180n - 360 = 174n - 2n^2$$

$$2n^2 + 6n - 360 = 0$$

$$n^2 + 3n - 180 = 0$$

$$(n+15)(n-12) = 0$$

Obviously $n = -15$ is an extraneous solution, and so we know that $n = 12$, and thus the polygon is a dodecagon.

Solution 2 by: Benjamin Lee (9/MD)

The sum of the exterior angles of the polygon is always 360°. So,

$$\text{Sum}_{exterior\ angles} = 8 + (8 + 4(1)) + (8 + 4(2)) + \cdots + (8 + 4(n-2)) + (8 + 4(n-1)) = 360,$$

where n is the number of sides of the polygon.

The equation can be simplified to

$$8n + 4(1 + 2 + \cdots + (n-2) + (n-1)) = 360$$
$$2n + \frac{(n-1)(n)}{2} = 90$$
$$4n + (n-1)(n) = 180$$
$$n^2 + 3n - 180 = 0$$
$$(n-12)(n+15) = 0$$

Therefore $n = 12$ or $n = -15$. Since n must be a positive integer, $n = 12$ and we are done.