
3/4/19. Let $0 < \mu < 1$. Define a sequence $\{a_n\}$ of real numbers by $a_1 = 1$ and for all integers $k \geq 1$,

$$\begin{aligned} a_{2k} &= \mu a_k, \\ a_{2k+1} &= (1-\mu) a_k. \end{aligned}$$

Find the value of the sum $\displaystyle\sum_{k=1}^{\infty} a_{2k} a_{2k+1}$ in terms of μ.

Credit This problem was proposed by Sandor Lehoczky, and modified by Dave Patrick.

Comments This following solution deftly finds the required sum by directly using the given recursion relations. *Solutions edited by Naoki Sato.*

Solution by: Tony Jin (10/CA)

By the definition of $\{a_n\}$,

$$\sum_{k=1}^{\infty} a_{2k} a_{2k+1} = \sum_{k=1}^{\infty} [\mu a_k \cdot (1-\mu) a_k] = \mu(1-\mu) \sum_{k=1}^{\infty} a_k^2.$$

We can split up the sum $\sum_{k=1}^{\infty} a_k^2$ as follows:

$$\begin{aligned} \sum_{k=1}^{\infty} a_k^2 &= a_1^2 + \sum_{k=1}^{\infty} a_{2k}^2 + \sum_{k=1}^{\infty} a_{2k+1}^2 \\ &= a_1^2 + \sum_{k=1}^{\infty} \mu^2 a_k^2 + \sum_{k=1}^{\infty} (1-\mu)^2 a_k^2 \\ &= a_1^2 + \mu^2 \sum_{k=1}^{\infty} a_k^2 + (1-\mu)^2 \sum_{k=1}^{\infty} a_k^2 \\ &= a_1^2 + [\mu^2 + (1-\mu)^2] \sum_{k=1}^{\infty} a_k^2. \end{aligned}$$

Therefore,

$$[1 - \mu^2 - (1-\mu)^2] \sum_{k=1}^{\infty} a_k^2 = a_1^2,$$

so

$$\sum_{k=1}^{\infty} a_k^2 = \frac{a_1^2}{1 - \mu^2 - (1-\mu)^2} = \frac{1}{2\mu(1-\mu)}.$$

Finally, the sum we seek is

$$\sum_{k=1}^{\infty} a_{2k}a_{2k+1} = \mu(1-\mu)\sum_{k=1}^{\infty} a_k^2 = \frac{1}{2}.$$