
3/2/18. The expression $\lfloor x \rfloor$ means the greatest integer that is smaller than or equal to x, and $\lceil x \rceil$ means the smallest integer that is greater than or equal to x. These functions are called the *floor function* and *ceiling function*, respectively. Find, with proof, a polynomial $f(n)$ equivalent to

$$\sum_{k=1}^{n^2} \left(\left\lfloor \sqrt{k} \right\rfloor + \left\lceil \sqrt{k} \right\rceil \right)$$

for all positive integers n.

Credit This problem was proposed by Scott Kominers, a past USAMTS participant.

Comments The first thing we want to do in this sum is remove the floor and ceiling notation. Since $\sqrt{k}$ is an integer when k is a perfect square, we can consider what happens when k lies between consecutive perfect squares. Once the floor and ceiling brackets have been removed, the rest of the problem is an exercise in algebra using standard summation formula. *Solutions edited by Naoki Sato.*

Solution 1 by: Shotaro Makisumi (11/CA)

Let m be a positive integer. For $(m-1)^2 + 1 \le k \le m^2 - 1$, we have $(m-1)^2 < k < m^2 \Rightarrow m - 1 < \sqrt{k} < m \Rightarrow \lfloor \sqrt{k} \rfloor + \lceil \sqrt{k} \rceil = (m-1) + m = 2m - 1$. For $k = m^2$, $\lfloor \sqrt{k} \rfloor + \lceil \sqrt{k} \rceil = m + m = 2m$. Hence,

$$\begin{aligned}
\sum_{k=(m-1)^2+1}^{m^2} (\lfloor \sqrt{k} \rfloor + \lceil \sqrt{k} \rceil) &= [(m^2 - 1) - (m-1)^2](2m-1) + 2m \\
&= (m^2 - 1 - m^2 + 2m - 1)(2m - 1) + 2m \\
&= (2m-2)(2m-1) + 2m \\
&= 4m^2 - 4m + 2,
\end{aligned}$$

which implies

$$\begin{aligned}
\sum_{k=1}^{n^2}(\lfloor\sqrt{k}\rfloor+\lceil\sqrt{k}\rceil) &= \sum_{m=1}^{n}\left[\sum_{k=(m-1)^2+1}^{m^2}(\lfloor\sqrt{k}\rfloor+\lceil\sqrt{k}\rceil)\right] \\
&= \sum_{m=1}^{n}(4m^2-4m+2) \\
&= 4\sum_{m=1}^{n}m^2 - 4\sum_{m=1}^{n}m + 2\sum_{m=1}^{n}1 \\
&= 4\cdot\frac{n(n+1)(2n+1)}{6} - 4\cdot\frac{n(n+1)}{2} + 2n \\
&= \frac{4(2n^3+3n^2+n)}{6} - \frac{12(n^2+n)}{6} + \frac{12n}{6} \\
&= \frac{8n^3+4n}{6} \\
&= \frac{4n^3+2n}{3}.
\end{aligned}$$

Therefore,

$$f(n) = \frac{4n^3+2n}{3}.$$