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Solutions to Problem 3/1/16
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3/1/16. Given that $5r + 4s + 3t + 6u = 100$, where $r \geq s \geq t \geq u \geq 0$ are real numbers, find, with proof, the maximum and minimum possible values of $r + s + t + u$.

Credit This problem was inspired by a similar problem posed 35 years ago in the first round of Hungary's Dániel Arany Mathematical Competition for students of advanced standing.

Comments Three elegant algebraic solutions are presented below. Some students also solved this problem by considering the region of 4–dimensional space described by the inequalities $r \geq s \geq t \geq u \geq 0$. The minimum and maximum of $r+s+t+u$ must located at the 'corners' of this space. Thus, we must test $(x, 0, 0, 0)$; $(x, x, 0, 0)$; $(x, x, x, 0)$; and (x, x, x, x) by finding the value of x in each case which satisfies the given $5r + 4s + 3t + 6u = 100$ and evaluating $r + s + t + u$ at the resulting points.

Solution 1 by: Yakov Berchenko-Kogan (10/NC)

Let:

$$\begin{array}{ccccccccc} u & + & a & & & & & = & t \\ u & + & a & + & b & & & = & s \\ u & + & a & + & b & + & c & = & r \end{array}$$

Since $r \geq s \geq t \geq u \geq 0$, we know $a, b, c \in \mathbb{R}_0^+$. Note that:

$$r + s + t + u = 4u + 3a + 2b + c$$

Substituting:

$$5r + 4s + 3t + 6u = 100$$

$$5(u + a + b + c) + 4(u + a + b) + 3(u + a) + 6u = 100$$

$$18u + 12a + 9b + 5c = 100$$

$$(2u + b + c) + 4(r + s + t + u) = 100$$

Clearly, in order to maximize $r+s+t+u$ we must minimize $2u+b+c$. Since all values are positive, this can easily be done by setting $u = b = c = 0$. Now, we can find what exactly the maximum value is:

$$4(r + s + t + u) = 100$$

$$r + s + t + u = 25$$

Thus 25 is the maximum value of $r + s + t + u$, achieved when $r = s = t = \frac{25}{3}$ and $u = 0$.

Now we must find the minimum value:

$$18u + 12a + 9b + 5c = 100$$

$$5(r + s + t + u) - (2u + 3a + b) = 100$$

Similarly to before, in order to minimize $r+s+t+u$ we must minimize $2u+3a+b$, and this is easily done by setting $u=a=b=0$. Again, we can easily find what exactly the minimum value is:

$$5(r+s+t+u) = 100$$

$$r+s+t+u = 20$$

Thus the minimum value of $r+s+t+u$ is 20, achieved when $r=20$ and $s=t=u=0$.

So, in summary, $20 \leq r+s+t+u \leq 25$.

Solution 2 by: Zachary Abel (11/TX)

Define $S = r+s+t+u$. Since $r \geq s \geq t \geq u \geq 0$, the numbers $r-s,\ s-t,\ t-u$, and u are non-negative. To find the lower bound, we calculate as follows:

$$\begin{aligned}
S &= r+s+t+u \\
&= (r-s)+2(s-t)+3(t-u)+4u \\
&\geq (r-s)+\tfrac{9}{5}(s-t)+\tfrac{12}{5}(t-u)+\tfrac{18}{5}u \\
&= \tfrac{1}{5}\big(5r+4s+3t+6u\big) \\
&= \tfrac{1}{5}\big(100\big) \\
&= 20.
\end{aligned}$$

The minimum of 20 can be achieved when $(r,s,t,u) = (20,0,0,0)$. We similarly find the upper bound:

$$\begin{aligned}
S &= r+s+t+u \\
&= (r-s)+2(s-t)+3(t-u)+4u \\
&\leq \tfrac{5}{4}(r-s)+\tfrac{9}{4}(s-t)+3(t-u)+\tfrac{9}{2}u \\
&= \tfrac{1}{4}\big(5r+4s+3t+6u\big) \\
&= \tfrac{1}{4}\big(100\big) \\
&= 25.
\end{aligned}$$

This maximum is attained when $(r,s,t,u) = (\frac{25}{3},\frac{25}{3},\frac{25}{3},0)$. Thus, the minimum and maximum values of S are 20 and 25 respectively.

Solution 3 by: Feiqi Jiang (9/MA)
Since $r \geq t$, we have $r - t \geq 0$. Also, $u \geq 0$ implies $2u \geq 0$. Adding this to $r - t \geq 0$ gives $r - t + 2u \geq 0$

Note that

$$4(r+s+t+u) + (r-t+2u) = 5r + 4s + 3t + 6u = 100.$$

Therefore,

$$\begin{aligned} 100 - 4(r+s+t+u) = (r-t+2u) &\geq 0 \\ 100 - 4(r+s+t+u) &\geq 0 \\ 100 &\geq 4(r+s+t+u) \\ 25 &\geq r+s+t+u \end{aligned}$$

Hence the maximum value of $r+s+t+u$ is 25.

We take a similar approach for the minimum: $s \geq u$ implies $s - u \geq 0$. Adding this to $2t \geq 0$ gives $s - u + 2t \geq 0$.

Note that

$$5(r+s+t+u) - (s-u+2t) = 5r + 4s + 3t + 6u = 100.$$

Therefore

$$\begin{aligned} 5(r+s+t+u) - 100 = s - u + 2t &\geq 0 \\ 5(r+s+t+u) - 100 &\geq 0 \\ 5(r+s+t+u) &\geq 100 \\ r+s+t+u &\geq 20 \end{aligned}$$

Thus the minimum value of $r+s+t+u$ is 20.