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1/2/19. Find the smallest positive integer n such that every possible coloring of the integers from 1 to n with each integer either red or blue has at least one arithmetic progression of three different integers of the same color.

Comments Any solution to this problem will inevitably require some casework. However, by choosing them carefully, the number of cases can be considerably reduced. *Solutions edited by Naoki Sato.*

Solution by: Adrian Chan (12/CA)

We first prove that $n \leq 8$ does not suffice. To do so, it is sufficient to give a counterexample for $n = 8$: Let 1, 4, 5, and 8 be red, and let 2, 3, 6, and 7 be blue. We see that there are no arithmetic sequences among the red numbers, or the blue numbers.

Now we show that for every coloring of the integers from 1 to 9, there is always an arithmetic sequence of three different integers of the same color. For the sake of contradiction, suppose that there is a coloring that does not produce any such arithmetic sequences. Without loss of generality, let 5 be blue. Then at least one of 1 and 9 must be red, otherwise 1, 5, and 9 will form a blue arithmetic sequence.

Case 1: 1 is blue and 9 is red, or 1 is red and 9 is blue.

First, assume that 1 is blue and 9 is red. Then 3 must be red, otherwise 1, 3, and 5 will form a blue arithmetic sequence. Next, 6 must be blue, otherwise 3, 6, and 9 will form a red arithmetic sequence. Next, both 4 and 7 must be red, otherwise 4, 5, and 6, or 5, 6, and 7 will form a blue arithmetic sequence. Finally, both 2 and 8 must be blue, otherwise 2, 3, and 4, or 7, 8, and 9 will form a red arithmetic sequence. However, we end up with 2, 5, and 8 forming a blue arithemtic sequence, contradiction. The case that 1 is red and 9 is blue can be similarly proven, by reversing the order of the colors.

Case 2: Both 1 and 9 are red.

First, assume that 7 is red. Then both 4 and 8 must be blue, otherwise 1, 4, and 7, or 7, 8, and 9 will form a red arithmetic sequence. Next, both 3 and 6 must be red, otherwise 3, 4, and 5, or 4, 5, 6 will form a blue arithmetic sequence. However, we end up with 3, 6, and 9 forming a red arithmetic sequence, contradiction.

Now, asssume that 7 is blue. Then 3 must be red, otherwise 3, 5, and 7 will form a blue arithmetic sequence. Next, 6 must be blue, otherwise 3, 6, and 9 will form a red arithmetic sequence. However, we end up with 5, 6, and 7 forming a blue arithmetic sequence, contradiction.

We conclude that $n = 9$ is the smallest possible integer that satisfies the desired conditions.